PG MA S1- 03 (DSC)

TWO YEAR MASTER'S DEGREE COURSE

SEMESTER - I

Real Analysis

MATHEMATICS

Discipline Specific Core Course
MASTER OF SCIENCE IN MATHEMATICS

Hiranya Chandra Bhuyan School of Science and Technology

KRISHNA KANTA HANDIQUI STATE OPEN UNIVERSITY

Subject Expert

Prof. Bipan Hazarika, Gauhati University

Prof. Nazibuddin Ahmed, Gauhati University

Prof. Dipak Sarma, Cotton University

Dr. Dilip Kumar, Cotton University

Course Co-ordinator : **Harekrishna Deka,** KKHSOU

SLM Preparation Team

UNITS	CONTRIBUTORS
1, 2, 3, 4, 5, 6, 7	Dr. Dilip Kumar, Cotton University, Guwahati
8, 9, 10, 11, 12, 13, 14	Dr. Binandita Barman, Goalpara College, Goalpara

Editorial Team

Content : **Harekrishna Deka,** KKHSOU

Structure, Format & Graphics : Harekrishna Deka, KKHSOU

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COURSE INTRODUCTION

"Real analysis" is the third Discipline Specific core course of MSc (Mathematics) programme. Real analysis is a foundational branch of mathematics that deals with the rigorous study of real numbers, sequences, series, limits, continuity, differentiation, integration, and more. It provides the theoretical underpinnings for calculus and forms the basis for many other areas of mathematics.This course is designed to introduce you to the key concepts and techniques of real analysis, building your understanding from the basics to more advanced topics. Each unit will focus on specific aspects of real analysis, providing theoretical insights, proofs, and examples to illustrate the concepts introduced.

Unit 1: Real Number System-I

We begin by examining the foundational properties of the real number system, including field axioms, absolute value, supremum, infimum, completeness axiom, and various properties of real numbers.

Unit 2: Real Number System-II

This unit explores concepts such as neighborhoods of points, limit points, open and closed sets, countability of sets, and Dedekind's property of real numbers.

Unit 3: Sequence

Sequences are a fundamental concept in real analysis. Here, we study boundedness, convergence, divergence, Cauchy sequences, limit superior and limit inferior, and more.

Unit 4: Infinite Series-I

This unit focuses on the convergence and divergence of infinite series, including Cauchy's general principle of convergence and auxiliary series.

Unit 5: Infinite Series-II

Further exploring convergence tests for series, this unit covers comparison tests, root and ratio tests, condensation test, logarithmic test, and alternating series.

Unit 6: Power Series

Power series are a powerful tool in mathematical analysis. We discuss their properties, radius of convergence, and applications.

Unit 7: Limits of Functions

This unit focuses on limits of functions, including right-hand and left-hand limits, infinite limits, and algebraic properties of limits.

Unit 8: Continuous Functions

Continuity is a central concept in real analysis. We explore different types of continuity, discontinuities, and properties of continuous functions.

Unit 9: Differentiable Functions

We delve into the theory of differentiation, including derivatives, mean value theorems, Taylor's theorem, and applications.

Unit 10: Functions of Several Variables

Extending our understanding to functions of multiple variables, we discuss limits, continuity, and partial derivatives.

Unit 11: The Riemann Integral

The Riemann integral is a key concept in calculus. Here, we explore partitions, Riemann sums, integrability, and the fundamental theorem of integral calculus.

Unit 12: The Riemann-Stieltjes integral

We generalize the Riemann integral to include integrals with respect to a differentiable function, studying partitions, integrability conditions, and relationships with the Riemann integral.

Unit 13: Uniform convergence of sequences and series

Uniform convergence of sequences and series is explored, along with its implications for continuity, integration, and differentiation.

Unit 14: Convergence of Improper integrals

Finally, we study convergence criteria for improper integrals and absolute convergence.

Each unit includes some along-side boxes to help you know some of the difficult, unseen terms. Some "EXERCISES" have been included to help you apply your own thoughts. You may find some boxes marked with: "LET US KNOW". These boxes will provide you with some additional interesting and relevant information. Again, you will get "CHECK YOUR PROGRESS" questions. These have been designed to self-check your progress of study. It will be helpful for you if you solve the problems put in these boxes immediately after you go through the sections of the units and then match your answers with "ANSWERS TO CHECK YOUR PROGRESS" given at the end of each unit.

UNIT 1 : REAL NUMBER SYSTEM- I

UNIT STRUCTURE

1.1 Learning Objectives

1.2 Introduction

1.3 Field axioms

1.4 Properties of Real Numbers

1.4.1 Algebraic Properties of Real Numbers.

1.4.2 Order properties of real Numbers

1.5 Absolute Value (or modulus) of a real Number

1.6 Completeness Property of the Real Numbers.

1.7 Archimedean Property of Real Numbers.

1.8 Density Property of Real Numbers.

1.9 Extended Set of Real Numbers

1.10 Let Us Sum Up

1.11 Answer to Check Your Progress

1.12 Further Reading

1.13 Model Questions

1.1 LEARNING OBJECTIVES

After going through this unit, you will able to know -

- field axioms, properties of real numbers, absolute value (or modulus of a real number
- completeness property of real numbers
- archimedean property of real numbers
- density property of real numbers.

1.2 INTRODUCTION

Real analysis is an indispensable branch of Mathematics. It deals with the study of real numbers. Generally calculus emphasizes learning the methods of differentiation and integration. In real analysis we study the principles that provide these methods.

In this unit, we will study field axioms, properties of real numbers, absolute value (or modulus) of a real number; completeness property of real numbers; Archimedean property of real numbers; density property of real numbers.

1.3 FIELD AXIOMS

Let S be a nonempty set. A rule, which associates every pair of elements of S with a unique element of S, is called a binary operation on S.

We know that the sum of two natural numbers is always a natural number. So, addition is a binary operation on the set N of natural numbers. Similarly multiplication is a binary operation on N, but subtraction and division are not binary operations on N.

A nonempty set F equipped with two binary operators defined on it, denoted by the symbols '+' (addition) and " . " (multiplication), is called a field if the following axioms are satisfied.

A1 : $a + b \in F, \forall\ a, b \in F$

i.e; F is closed under "+"

A2 : $(a + b) + c = a + (b + c) \ \forall\ a, b, c, \in F$,

i.e; "+" is associative

A3 : There exists an element 0 in F, called the zero or the zero element of F such that $a + 0 = a = 0 + a, \ \forall\ a \in F$

A4 : For every $a \in F$, there exists an element $-a \in F$ such that

$a + (-a) = 0 = (-a) + a$

–a is called addivitive inverse of a

A5 : $a + b = b + a, \ \forall\ a, b \in F$

i.e; "+" is commutative

M1 : $a. b \in F, \ \forall\ a, b \in F$

i.e., F is closed undder "."

M2 : $(a . b). c = a. (b . c)$

i.e., " . " is associative.

M3 : There exists an element 1 in F different from 0, called the unity of F suc that

a.1 = a = 1.a, $\forall$ a $\in$ F

M4 : For every a $\in$ F, a $\neq$ 0, there exists an element $a^{-1} \in$ F such that

$a.a^{-1} = a = a^{-1}.a$

a^{-1} is called multiplicative inverse of a

M5 : a . b = b. a, $\forall$ a, b $\in$ F

i.e., " . " is commutative

D : a. (a + c) = a. b + a. c, (left distributive law0

(a + b). c = a, c + b. c, (right distributive law)

$\forall$ a, b, c $\in$F

Note

1. A nonempty set, equipped with one or more binary operations defined on it, is called an algebraic structure. Clealy a field is an algebraic structure.
2. In a field unity is not equal zero, i.e., 1 $\neq$ 0. So, a field must have at least two elements.
3. It is clear from the definition of binary operation that F is closed under '' + '' and " . ". So, the axioms A1 and M1 are superfluous.
4. In a field, additive inverse as well as multiplicative inverse of an element is unique.
5. In a field, the zero and the unity are unique.

Example of Field

1. The set $\underline{\underline{Q}}$ of rational numbers is a field under ordinary addition and multiplication as the eleven axioms of a field are satisfied.
2. The set of $\underline{\underline{R}}$ of real numbers is a field under ordinary addition and multiplication.

Note :

The set $\underline{\underline{N}}$ of natural numbers is not a field. We have 5 $\in \underline{\underline{N}}$ but – 5 $\notin \underline{\underline{N}}$

The set $\underline{\underline{Z}}$ of integers is not a field. We have $5 \in \underline{\underline{Z}}$, but $\frac{1}{5} \notin \underline{\underline{Z}}$.

Remark : In the subsequent discussions, we will omit the symbol ".". We will write ab instead of a.b.

1.4 PROPERTIES OF REAL NUMBERS

1.4.1 Algebraic properties of real numbers

Theorem 1.1 Let a, b, c be real numbers. Then

(i) a + b = a + c implies b = c

(Left cancellation law for addition)

(ii) ab = ac implies b = c, $a \neq 0$

(Left cancellation law for multiplication)

Proof :

(i) We have $a \in \underline{\underline{R}}$

$\therefore -a \in \underline{\underline{R}}$

a + b = a + c

$\Rightarrow$ (–a) + (a + b) = (–a) + (a + c)

$\Rightarrow$ (– a + a) + b = (– a + a) + c, by A2

$\Rightarrow$ 0 + b = 0 + c, by A4

$\Rightarrow$ b = c

(ii) We have $a \in \underline{\underline{R}}$, $a \neq 0$

$\therefore a^1 \in \underline{\underline{R}}$

ab = ac

$\Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$, by M2

$\Rightarrow$ 1b = 1c, by M4

$\Rightarrow$ b = c

Remark : Similarly we can prove that

(iii) b + a = c + a implies b = c

(Right cancellation law for addition)

(iv) ba = ca implies b = c, $a \neq 0$

(Right cancellation law for multiplication)

Theorem 1. 2 Let a be any real

Then (i) a0 = 0 = 0a

(ii) (–1) a = – a = a (–1)

(iii) – (–a) = a

(iv) $(a^{-1}) = a, a \neq 0$

Proof : (i) We have $0 + 0 = 0$

$\Rightarrow a (0 + 0) = a0$

$\Rightarrow a0 + a0 = a0$, by D

$\Rightarrow a0 + a0 = a0 + 0$

$\Rightarrow a\ 0 = 0$

(by left cancellation law for addition

similarly $0a = 0$

(ii) We have $1+(-1)=0$

$\Rightarrow (1 + (-1)\ a = 0\ a$

$\Rightarrow (1\ (-1))\ a = 0$

$\Rightarrow 1a + (-1)\ a = 0$

$\Rightarrow a + (-1)\ a = 0$

$\Rightarrow (-a) + (a + (-1)\ a) = (-a) + 0, [\ a \in \underline{\underline{R}} \Rightarrow -a \in \underline{\underline{R}}\ ]$

$\Rightarrow ((-a) + a) + (-1)\ a = -a$

$\Rightarrow \quad 0 + (-1)\ a = -a$

$\Rightarrow \quad (-1)a = -\ a$

similarly, We can prove that

$a\ (-1) = -\ a$

(iii) We have $a + (-a) = 0$, by A4 ... (1)

Replacing a by $-\ a$, we get

$-\ a + (-(-a)) = 0$ ... (2)

From (1) and (2)

$-\ a + (-(-a)) = a + (-a)$

$\Rightarrow -\ a + (-(-a)) = (-a) + a$, by A5

$\Rightarrow (-\ (-a)) = a,$

by lefy cancellation law for addition.

(iv) We have $aa^{-1} = 1, a \neq 0$ by M4 ... (1)

Replacing a by a^{-1}, We get

$a^{-1}\ (a^{-1})^{-1} = 1$ ... (2)

From (1) and (2)

$a^{-1}\ (a^{-1})^{-1} =\ a\ a^{-1}$, by M5

$\Rightarrow a^{-1}(a^{-1})^{-1} = a^{-1}a$

$\Rightarrow (a^{-1})^{-1} = a$

by left cancellation law for multiplication.

Theorem 1.3 Let a, b be real numbers then

(i) $a(-b) = -(ab) = (-a)b$

(ii) $(-a)(-b) = ab$

Proof (i) We have $b + (-b) = 0$, by A4

$\therefore\ a(b + (-b) = a0$

$\Rightarrow ab + a(-b) = 0$, ny D and Theorem 1.2 ... (1)

Again $-(ab)$ is a real number

$\therefore\ -(ab) + (ab + a(-b)) = -(ab) + 0$

$\Rightarrow (-(ab) + (ab)) + a(-b) = -(ab)$, by A2, A3

$\Rightarrow 0 + a(-b) = -(ab)$, by A4

$\Rightarrow a(-b) = -(ab)$, by A3

Similarly $(-a)b = -(ab)$

(ii) We have $a(-b) = -(ab)$

Replacing a by (–a), We get

$(-a)(-b) = -((-a)b)$

$\Rightarrow (-a)(-b) = -(-(ab))$, Theorem 1.3 (i)

$\Rightarrow (-a)(-b) = ab$, Theorem1.2(iii)

1.4.2 Order properties of real numbers

We know that every real number can be represented by a unique point on line, and, conversely, every point on a line represents a unique real number.

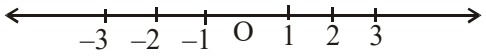


Fig 1.1

A real number a $\neq$ 0 said to be positive if it lies on the right side of the point 0 which corresponds to the real number zero. We write it as $a > 0$. Similarly a real number a is said to be negative if it lies on the left side of 0. We write it as $a < 0$

If $a \geq 0$ thew a is said to be nonnegative. Similarly a is said to be nonpositive if $a \leq 0$.

Let a and b be any two real numbers.

We say that a is greater than b if a – b > 0. We write it as a > b.

Similarly a is less than b if a – b < 0. We write it as a < b.

a is greater than or equal to b if a – b $\geq$ 0. We write it as a $\geq$ b.

Similarly a is less than or equal to b if a – b $\leq$ 0. We write it as a $\leq$ b.

The order relation < (0r >) and $\underline{\underline{R}}$ satisfies the following conditions.

01. Trichotomy Law

Let a, b be two real numbers. Then exactly one of the following holds.

(i) a is greater than (a > b).

(ii) a is equal to be (a = b)

(iii) a is less than b (a < b)

0.2 Order is transitive

Let a, b, c be real numbers such that a < b, and b < c. Then acc

03. Order is compatible with addition

a < b $\Rightarrow$ a + c < b + c, for any real numbers a, b, c.

04. ordering is compatible with multiplication

(i) a < b and c > 0 $\Rightarrow$ ac < bc

(ii) a < b and c < 0 $\Rightarrow$ ac < bc.

The field of real numbers $\underline{\underline{R}}$ together with the order relation defined on $\underline{\underline{R}}$ satisfying 01, 02, 03, 04 becomes an ordered field.

Note 1. The field of rational numbers $\underline{\underline{Q}}$ is ordered field.

2. The set $\underline{\underline{C}}$ of complex numbers is a field, but it is not ordered.

Order cannot be defined on $\underline{\underline{C}}$.

i = $\sqrt{-1}$ is neither positive nor negative

i $\neq$ 0, i is not a real number.

Do You Know?

The meaning of the word trichotomy is 'division into three categories'. In theology, it means 'the division of human beings into body, spirit, and soul.'

Source : Collins English Dictionary

Theorem 1.4 Let a, b be real numbers. Then

(i) $a > 0 \Rightarrow -a < 0$

(ii) $a < 0 \Rightarrow -a > 0$

(iii) $a > 0, b > 0 \Rightarrow a + b > 0$

(iv) $a < 0, b < 0 \Rightarrow a + b < 0$

(v) $a > 0, b > 0 \Rightarrow ab > 0$

(vi) $a < 0, b < 0 \Rightarrow ab > 0$

(vii) $a > 0, b < 0 \Rightarrow ab < 0$

Proof

(i) We have $a \in \underline{\underline{R}}$ and a>0

Now $a > 0, a + (-a) = 0$

By the law of trichotomy, $-a < 0$ or $-a=0$, or $-a > 0$

If possible, let $-a > 0$

$\therefore (-a) + a > 0 + a$, by 03

or $0 > a$,

Which is a contradication.

Again a = 0 is a contradiction.

$\therefore -a < 0$

(ii) Proceed as above

(iii) We have

$a > 0$

$\therefore a + b > 0 + b$, by 03

$\Rightarrow a + b > b$

Now $a + b > b$, and $b > 0$

$\therefore$ by 02, $a + b > 0$

(iv) Proceed as above

(v) We have $a > 0, b > 0$

$\therefore ab > ob$ by 04 (i)

$\Rightarrow ab > 0$

(vi) We have $a < 0, b < 0$

$\therefore a < 0, -b > 0$

$\Rightarrow$ $a(-b) < 0(-b)$, by 04 (i)

$\Rightarrow$ $a(-b) < 0(-b)$, by 04 (i)

$\Rightarrow$ – ab < 0

$\Rightarrow$ – (–ab) > 0, by (iii)

$\Rightarrow$ ab < 0

(vii) Try yourself.

Intervals

Let a, b be real numbers.

(i) The set {x : a $\leq$ x $\leq$ b} denoted by [a, b], is called a closed interval. The end points a, b are included.

(ii) The set {x : a < x < b}, denoted by]a, b[, is called an open interval. The end point a, b are not included.

(iii) The set {x : a < x $\leq$ b} is denoted by]a, b].

The set {x : a $\leq$ x < b } is denoted by [a, b[.

These two are called semi-clossed or semi-open intervals.

The former is open at a, and closed at b; while the latter is closed at a and open at b.

1.5 ABSOLUTE VALUE (OR MODULUS) OF A REAL NUMBER

Let x be any real number.

The absolute value (or the modulus) of x denoted by |x|, is defned by

$$|x| = \begin{cases} x, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -x, \text{if } x < 0 \end{cases}$$

For example |10| = 10, |0| = 0, |–10| = – (– 10) = 10

$|x| \geq 0$, for all $x \in \underline{\underline{R}}$

Note :

1. |x| is a nonnegative real number.

2. |–x| $\neq$ – |x|

3. |x| = 0 if and only if x = 0

Properties

1. If x is any real number,

|x| = max {x, – x}, where max {x, –x} denotes the greater of the two numbers x, – x.

Proof : x is any real number.

Therefore, there are three possibilities :

$x > 0$, or $x = 0$, or $x < 0$

case 1. Let $x > 0$

$\therefore \quad -x < 0$

$x > -x$

since $x > 0$, $|x| = x$

Maximum of $\{x , -x\} = x$

$\therefore |x| = \max (x, -x\}$

case 2 Let $x = 0$

$\therefore \quad x = 0 = -x$

$|x| = 0$

maximum of $\{x , -x\} = 0$

$\therefore |x| = \max \{x, -x\}$

case 3. Let $x < 0$

$\therefore \quad -x > 0$

$\therefore \quad -x > x$

Since $x < 0$, $|x| = -x$

Maxmimum of $\{x, -x\} = -x$

$\therefore |x| = \max \{x, -x\}$

Thus, for all $x \in \underline{\underline{R}}$,

$|x| = \max \{x, -x\}$

Note : It follows that $x \leq |x|, -x \leq ||x||$

2. If x is any real number, $|x| = |-x|$

Proof : We have $|x| = \max \{x, -x)$

$\therefore |x| = \max \{ -x, -(-x)\}$

$= \max \{-x, x\}$

$= \max \{x, -x\}$

$= |x|.$

3. If x is any real number, then $|x|^2 = x^2 = |-x|^2$

Proof: case 1. Let $x > 0$

$\therefore |x| = x$

$\therefore |x|^2 = |x|\,|x| = xx = x^2$

case 2. Let x = 0

$\therefore |x| = 0$

clearly $|x|^2 = x^2$

case 3. Let $x < 0$

$\therefore |x| = -x$

$\therefore |x|^2 = |x|\,|x| = (-x)(-x) = x^2$

Thus for all $x \in \underline{\underline{R}}$,

$|x|^2 = x^2 = |-x|^2$

4. If x, and y are any two real numbers, then

$|x < t| \leq |x| + |y|$

[The above inequality is known as the triangle inequality.]

Proof : x and y are any two real numbers.

$\therefore$ x + y is also a real number.

Therefore, there are three possibities, x + y > 0, or, x + y = 0, or, x + y < 0.

Let $x + y > 0$

$\therefore$ by definition $|x + y| = x + y$ ...(1)

Again, we know that

$x \leq |x|$, for all $x \in \underline{\underline{R}}$,

$y \leq |y|$, for all $y \in \underline{\underline{R}}$,

$\therefore$ $x + y \leq |x| + |y|$ ...(2)

For (1) and (2), if follows that

$|x + y| \leq |x| + |y|$

Let, x + y = 0

Then, by definition |x + y | = 0

$\therefore |x + y| = x + y$

Let $x + y < 0$

$\therefore$ by definition $|x + y| = -(x + y)$

$= (-x) + (-y)$ ...(3)

Again we know that

$-x \leq |x|$, for al $x \in \underline{\underline{R}}$

$-y \leq |y|$, for all $y \in \underline{\underline{R}}$

$\therefore$ $(-x) + (-y) \leq |x| + |y|$ ... (4)

From (3) and (4), it folowing that

$|x + y| \leq |x| + |y|$.

Thus for all x, $y \in \underline{\underline{R}}$, $|x + y| \leq |x| + |y|$

5. If x and y are any two real numbers,

Then $|x - y| \geq ||x| - |y||$

Proof We have

$x = (x - y) + y$

$\therefore |x| = |(x - y) + y|$

$\leq |x - y| + |y|$, by triangle inequality

$\therefore \quad |x| - |y| \leq |x - y|$ ... (1)

Interchanging x and y in (1), we get

$|y| - |x| \leq |y - x|$

Now $|y - x| = |-(x-y)| = |x - y|$

$\therefore \quad |y| - |x| \leq |x - y|$

or $\quad -(|x| - |y|) \leq |x - y|$ ... (2)

From (1) and (2), it folows that $|x| - |y|$ and its negative $-(|x| - |y|)$ are both less than or at most equal to $|x - y|$

$\therefore \max \{|x| - |y|, -(|x| - |y|)\} \leq |x - y|$

This implies $||x| - |y|| \leq |x - y|$,

i.e., $|x - y| \geq ||x| - |y||$

Thus, for all x. $y \in \underline{\underline{R}}$,

$|x\ y| \geq ||x| - |y||$

6. It x and y are any two real numbers, then

$|xy| = |x|\,|y|$

Proof : x and y are any two real numbers.

There are three possibities,

$x \geq 0, y \geq 0$

Or

x and y have opposite signs

Or

$x \leq 0, y \leq 0$

Let $x \geq 0, y \geq 0$

$\therefore \quad |x| = x, |y| = y$

$\therefore$ $x \geq 0, y \geq 0 \therefore xy \geq 0$

$\therefore$ $|xy| = xy = |x|\,|y|$

Let x and y have opposite signs.

Let $x > 0\ y < 0$

$\therefore$ $|x| = \max\{x, -x\} = x$

$\therefore$ $|y| = \max\{y, -x\} = -y$

$\because$ $x > 0, y < 0 \therefore xy < 0$

$\therefore$ $|xy| = -(xy) = x(-y) = |x|\,|y|$

Let $x \leq 0, y \leq 0$

$\therefore$ $|x| = -x, |y| = -y$

$\therefore$ $x \geq 0, y \leq 0 \therefore xy \geq 0$

$\therefore$ $|xy| = xy = (-x)(-y) = |x|\,|y|$

Thus, for all $x, y \in \underline{\underline{R}}$,

$|xy| = |x|\,|y|$

7. If x is any non zero real number, then $\left|\frac{1}{x}\right| = \frac{1}{|x|}$

Proof : Let $x > 0$

$\therefore |x| = x$

Also, $\frac{1}{x} > 0$

$\therefore$ $\left|\frac{1}{x}\right| = \frac{1}{x} = \frac{1}{|x|}$

Let $x < 0$

Also $\frac{1}{x} < 0$

$\therefore$ $\left|\frac{1}{x}\right| = -\frac{1}{x} = \frac{1}{|-x|} = \frac{1}{|x|}$

Thus, for any nonzero real number x,

$$\left|\frac{1}{x}\right| = \frac{1}{|x|}$$

8. If x is a real number and y is a nonzero real number, then

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

Proof :

$$\left|\frac{x}{y}\right| = \left|x.\frac{1}{y}\right|$$

$$= |x| \left|\frac{1}{y}\right| \text{ (Property 6)}$$

$$= |x| \left|\frac{1}{|y|}\right| \text{ (Property 7)}$$

$$= \frac{|x|}{|y|}$$

9. If x and y are any two real numbers, then

$|x + y|^2 + |x – y|^2 = 2 (|x|^2 + |y|^2)$

Proof: We have

$|x + y|^2 = (x + y)^2$, (Property 3)

$|x – y|^2 = (x – y)^2$, (roperty 3)

$\therefore \quad |x + y|^2 + |x – y|^2 = (x + y)^2 + (x – y)^2$

$= 2 (x^2 + y^2)$

$= 2 (|x|^2 + |y|^2)$

10. If x and y are any two real numbers and $\delta > 0$ then

$|x – a| < \delta$ if and only $a – \delta < x < a + \delta$

Proof: We have

$|x – a| = \max \{(x – a), – (x – a)\}$

$= \max \{(x – a), a – x)\}$

Now, $|x – a| < \delta$

$\Leftrightarrow \max \{(x – a), (a – x)\} < \delta$

$\Leftrightarrow x – a < \delta$ and $a – x < \delta$

$\Leftrightarrow x < a + \delta$ and $a – \delta < x$

$\Leftrightarrow a – \delta < x < a + \delta$

11. If x is any real number, and $\delta < 0$, then $|x| < \delta$ if and only if $– < \delta < x < \delta$

Proof: We have $|x| = \max \{x , – x\}$

$|x| < \delta$

$\Leftrightarrow \max \{(x – x) < \delta$

$\Leftrightarrow x < \delta$ and $– x < \delta$

$\Leftrightarrow x < \delta$ and $– \delta < x$

$\Leftrightarrow -\delta < x < \delta$

Example 1. Let x and y be two real numbers. Show that

$|x + y| = |x| + |y|$

if and only if $xy \geq 0$

Solution : $|x + y| = |x| = |y|$

$\Leftrightarrow |x + y|^2 = (|x| + |y|)^2$

$\Leftrightarrow |x + y|^2 = (|x| + |y|)^2$

$\Leftrightarrow (x + y)^2 = |x|^2 + 2|x|\,|y| + |y|^2$, $[|x|^2 = x^2]$

$\Leftrightarrow x^2 + 2xy + y^2 = x^2 + 2|xy| + y^2$

$\Leftrightarrow xy = |xy|$

$\Leftrightarrow xy \geq 0$, by definition.

Example 2. Write the inequality $6 < x < 8$ in the absolute value form.

Solution : We know that $|x - a| < \delta$ if and only if $x - \delta < x < a + \delta$.

$6 < x < 8$ (given)

$\Rightarrow 7 - 1 < x < 7 + 1$

$\Rightarrow |x - 7| < 1$

Example 3. Rewrite the inequality $|x - 4| < 1$ by removing the absolute value symbol.

Solurion : We know that $|x - a| < \delta$ if and only if $a - \delta < x < a + \delta$

$|x - 4| > 1$

$\Rightarrow 4 - 1 < x < 4 + 1$

$\Rightarrow 3 < x < 5$

Example 4. If in the inequality $|x - a| < \delta$, $a = 5$, $\delta = 0.01$, write down any three values of x satisfying $|x - a| < \delta$

Solution : We know that $|x - a| < \delta$ if and only if $a - \delta < x < a + \delta$

Here $a = 5$, $\delta = 0.01$

$\therefore |x - 5| < .01$

$\Rightarrow 5 - 0.01 < x < 5 + 0.01$

$\Rightarrow 4.99 < x < 5.01$

$\therefore$ any three values of x satisfting the above inequality are 4.999, 5, 5.001.

Example 5. Solve $\left|\dfrac{x+2}{x-6}\right| = 5$

Solution : $\left|\frac{x+2}{x-6}\right| = 5$ given

$\therefore \left|\frac{x+2}{x-6}\right| = 5$ or $\left|\frac{x+2}{x-6}\right| = -5$

Now $\left|\frac{x+2}{x-6}\right| = 5 \quad \Rightarrow x + 2 = 5 (x - 6)$

$\Rightarrow x + 2 = 5x - 30$

$\Rightarrow 4x = 32$

$\therefore x = 8$

Again $\left|\frac{x+2}{x-6}\right| = -5 \quad \Rightarrow x + 2 = -5 (x - 6)$

$\Rightarrow x + 2 = -5x + 30$

$\Rightarrow 6x = 28$

$\therefore x = \frac{28}{2} = \frac{14}{3}$

$\therefore \quad x = 8, \frac{14}{3}$

Absoluts Value Function

Using the concept of absolute value of a real number, the absolute value function (or the modulus function) can be defined as follows

$$f(x) = |x| = \begin{cases} x, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -x, \text{if } x < 0 \end{cases}$$

The graph of the function is shown in Fig 1.2

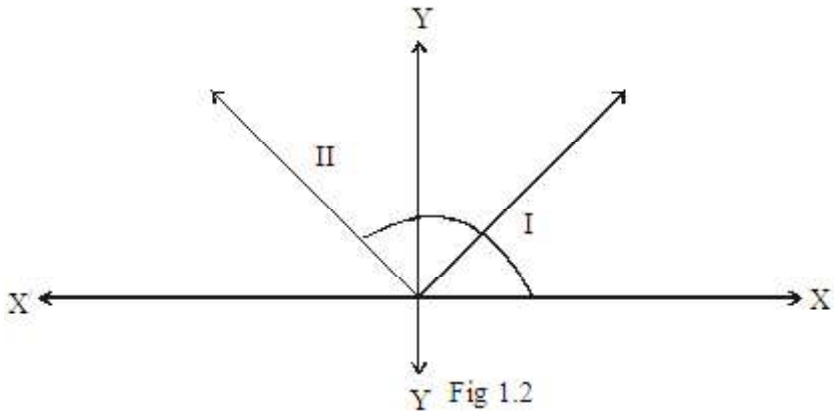


Fig 1.2

Note : 1. The graph consists of two rays, both starting at the origin.

2. The ray I marks an angle of $\frac{\pi}{4}$ with the positive direction of the x–

axis. The ray II markes an angle of $\frac{3\pi}{4}$ with the positive direction of the x - axis. The two rays are perpendicular to each other.



CHECK YOUR PROGRESS

Q.1. If x, y, z are any three real numbers, prove that

(i) $|x + y + z| \leq |x| + |y| + |x|$

(ii) $|xyz| = |x|\,|y|\,|z|$

Note : The above results can be extended to any finite number of real numbers. Thus if $x_1, x_2, x_3 \ldots, x_4$ are any real numbers then

(i) $|x_1 + x_2 + x_3 + \ldots\ldots + x_4| \leq |x_1| + |x_2| + |x_3| + \ldots\ldots + |x_4|$

(i) $|x_1\ x_2\ x_3 \ldots\ldots x_4| = |x_1|\ |x_2|\ |x_3|\ \ldots\ldots |x_4|]$

Q. 2. If in the inequality $|x - a| < \delta$, a = 9, δ =0.002, write down any three values of x satisfying $|x - a| < \delta$.

Q. 3. Write the inequality $10 < x < 12$ in the absolute value form.

Q. 4. Rewrite the inequality $|x - 7| < 0.0001$ by removing the absolute value symbol.

Q. 5. If x, y are any two real numbers, prove that $|x - y| \leq |x| + |y|$

Q.6. Show that $a < x < b$ if and only if $\left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$

Q. 7. Show that for any two real numbers a and b.

(i) $\max\{a, b\} = \frac{1}{2}[a + b + |a - b|]$

(ii) $\min\{a , b\} = \frac{1}{2}[a + b - |a - b|]$

Q. 8. Solve $\left|\frac{x+5}{3x-1}\right| = 4$

Q. 9. Show that $|a + b| = |a| + b|$ if and only if $ab \geq 0$.

Q.10. Show that $|a + b| < |a| + |b|$ if and only $ab < 0$

Q. 11. Which of the following statements are true ?

(i) $\{x : |x - 5| < 2\} = \{x : 3 < x < 7\}$

(ii) $\{x : |x - 4| < 1\} = \{x : -1 < x < 1\}$

(iii) $|x : |x - 1| < 2\} = \{x : 1 < x < 3\}$.

Q.12. If $f(x) = \frac{|x|}{x}$, and a is a nonzero real number, show that $|f(a) - f(-a)| = 2$

Q.13. If a and b are real numbers, show that

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

Q.14. If a and b are real numbers, show that

$$\frac{|a-b|}{1+|a-b|} = \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

1.6 COMPLETENESS PROPERTY OF THE REAL NUMBERS

Let us consider the set

$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots\right\}$$

We see that each element of the set S is greater than zero. Then we say that 0 is a lower bound of S. Similary –1, – 2, etc. are also lower bounds of S.

Again, we see that each element of S in less than or equal to one. We say that 1 is an upper bound of S. Similarly 2, 3, etc. are upper bounds of S

Again, we see that 0 is the greatest of all lower bounds and 1 is the least of al upper bounds.

Definitions : Let S be a subset of $\underline{\underline{R}}$.

(i) S is bounded above if there exists $u \in \underline{\underline{R}}$ such that $x \leq u$ for all $x \in S$. u is an upper bound of S.

(ii) S if bounded below if there exists $w \in \underline{\underline{R}}$ such that $w \leq x$ for all $x \in S$. w is a lower bound of S.

(iii) S is bounded if it is both bounded above and bounded below.

(iv) S is unbounded if it is not bounded.

(v) Let S be bounded above. An upper bound u of S is said to be least upper bound (or supremum) of S if $u \leq u'$ for any upper bound of S. Symbolically we write u =sup(S) = lub (S).

(vi) Let S be bounded below. A lower bound w of S is said to be greatest lower bound (or infimum) of S if $w' \leq w$ for any lower bound w' of S.

Symblically we write w = inf (S) = glb (S)

Note : Any finite set is bounded

Examples (i) Let us consider the set

$$S = \left\{ \frac{n}{n+1} : n \in \underline{\underline{N}} \right\}$$

$$= \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ... \right\}$$

Clearly S is bounded

lub (S) = sup(S) = 1

glb (S) = inf (S) = $\frac{1}{2}$

(ii) Let us consider the intervals [a, b],]a, b[,]a, b[

Each of these intervals is bounded

lub = a, glb = b

(iii) The set { $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$, ...} is bounded.

(iv) The set $\underline{\underline{N}}$ of natural numbers is bounded below, but not bounded above.

(v) The set $\underline{\underline{Z}}$ of integers is unbounded.

(vi) The set $\underline{\underline{Q}}$ of rational numbers is unbounded.

(vii) The set $\underline{\underline{R}}$ of real numbers is unbounded.

An empty set is bounded. Every real number x is an upper bound. Every real number x is a lower bound.

Example 6 : Let $A \subseteq \underline{\underline{R}}$, B $\subseteq \underline{\underline{R}}$ such that $A \subseteq B$.

(i) prove that if B is bounded above (below), then A in bounded above (below).

(ii) Give an example of sets A and B such that A is bounded below, but B is not bounded below.

Solution (i) Let B be bounded above

$\therefore$ there exists $u \in \underline{\underline{R}}$ such that $x \leq u$, for all $x \in B$

$\therefore$ $y \leq u$, for all $y \in A$.

$\therefore$ A is bounded above.

Similarly, it can be shown that if B is bounded below, then A is bounded below.

(ii) Let A = $\underline{\underline{N}}$, the set of natural numbers, and $B = \underline{\underline{Z}}$, the set of integers clearly, $A \subseteq B$.

Moreover A is bounded below, but B is not bounded.

Theorem 1.5: (i) Union of two bounded sets is bounded.

(ii) Intersection of two bounded sets is bounded.

Proof : (i) Let $A \subseteq \underline{\underline{R}}$, $B \subseteq \underline{\underline{R}}$ such that A. B are bounded.

A is bounded $\Rightarrow$ A is bounded above, and bounded below.

B is bounded $\Rightarrow$ B is bounded above, and bounded below.

$\therefore$ There exist w_1 (lower bound) and u_1 (upper bound) such that

$w_1 \leq x \leq u_1$, for all $x \in A$.

Again, there exist w_2 (lower bound) and u_2 (upper bound) such that

$w_2 \leq x \leq u_2$, for all $x \in B$

Now $x \in AUB$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow w_1 \leq x \leq u_1$ or $w_2 \leq x \leq u_2$

We choose w = minimum (w_1, w_2)

u = maximum (u_1, u_2)

Thus $x \in AUB$

$\Rightarrow w \leq x \leq u$

$\Rightarrow$ AUB is bounded.

(ii) Try yourself.

Theorem 1.6 (i) The supremum of a set, if it exists, is unique

(ii) The infimum of a set, if it exists is unique.

Proof : (i) Let $A \subseteq \underline{\underline{R}}$

If possible, let there be two suprema, say, u_1 and u_2 of A.

$\because$ u_1 is supremum, by definition,

$u_1 \leq u_2$ ... (1)

Again, $\because$ u_2 is supremum, by definition, $u_2 \leq u_1$ ... (2)

From (1) and (2), $u_1 = u_2$

Thus the supremum is unique

(ii) Try yourself

Note : 1. If a set is bounded above, there exist infinitely many upper bounds. But the least upper bound (supremum) is unique.

2. If a set is bounded below, there exist infinitely many lower bounds. But the greatest lower bound (infimum) is unique.

Do you Know?

According to The Chambers Dictionary an axiom is a self-evident truth, a universally received principle an assumption made for the purpose of argument.

Completeness axiom (or the supremum axiom)

Every nonempty subset of $\underline{\underline{R}}$ that is bounded above has the least upper bound (supremum) in $\underline{\underline{R}}$.

Note : Every nonmepty subset of $\underline{\underline{R}}$ that is bounded below has the greatest lower bound (infimum) in $\underline{\underline{R}}$. It can be proved using the completeness axiom.

Theorem 1.7: A nonempty subset of $\underline{\underline{R}}$ that is bounded below has the greatest lower bound in $\underline{\underline{R}}$

Proof Let S be a nonempty subset of $\underline{\underline{R}}$ that is bounded below.

Let w be a lower bound of S.

$\therefore$ $w \leq x, \forall x \in S$

Let $T = \{-x : x \in S\}$

$\because$ $S \neq \phi$, $\therefore T \neq \phi$.

We have $w \leq x, \forall x \in S$

$\Rightarrow -w \geq -x, \forall x \in S$

$\Rightarrow -x \leq -w, \forall x \in S$

$\therefore$ T is bounded above with – x as an upper bound.

Thus, T is nonempty and it is bounded above.

$\therefore$ by completeness axiom.

T has two least upper bound is $\underline{\underline{R}}$

Let it be M

$\therefore -x \leq M, \forall x \in S$

$\Rightarrow x \geq -M, \forall x \in S$

$\Rightarrow -M \leq x, \forall x \in S$

$\Rightarrow$ – M is a lower bound of S.

Let w' be any lower bound of S.

$\therefore$ – w' is an upper bound of T.

$\therefore$ M = lub (T), $\therefore M \leq -w'$

$\Rightarrow -M \geq w'$

$\Rightarrow w' \leq -M$

$\therefore -M = \text{glb (S)}$.

Thus, S has the greatest lower bound in $\underline{\underline{R}}$

Definition : An ordered field is said to be complete ordered field if the completeness axiom holds in it.

Thus, $\underline{\underline{R}}$ is a complete ordered field.

Note : The property of completeness does not hold good for the ordered field of rational numbers.

Example 7. Let the subset S of $\underline{\underline{Q}}$ be defined by

S = { $x \in \underline{\underline{Q}}$: x > 0 and $x^2 < 2$} Show that S is a nonempty subset of $\underline{\underline{Q}}$ bounded does not belong to $\underline{\underline{Q}}$.

Solution : We have 1 > 0, and $1^2 < 2$

$\therefore 1 \in S$, i.e., $S \neq \phi$.

Clearly 2 is an upper bound S.

$\therefore$ S is bounded above.

If possible, let lub (S) $\in \underline{\underline{Q}}$.

Let lub (S) = u

Then u > 0, and $u \in \underline{\underline{Q}}$.

By the law of trichotomy, there are three possibilitries,

$u^2 < 2$ or $u^2 = 2$ or $u^2 > 2$

Case1. Let $u^2 < 2$

We define a rational number r by

$$r = \frac{4+3u}{3+2u}$$

Clearly r > 0, $\because u > 0$

Again $r - u = \frac{4+3u}{3+2u} - u$

$$= \frac{4+3u-u(3+2u)}{3+2u} = \frac{2(2-u^2)}{3+2u}$$

$\therefore u^2 < 2$, $\therefore 2 > u^2$, i.e $2 - u^2 > 0$

$$\therefore r - u = \frac{2(2-u^2)}{3+2u} > 0$$

i.e., r > u

Again $r^2 - 2 = \left(\frac{4+3u}{3+2u}\right)^2 - 2$

$$= \frac{(4+3u)^2 - 2(3+2u)^2}{(3+2u)^2}$$

$$= \frac{u^2 - 2}{(3+2u)^2}$$

$\because u^2 < 2, \therefore u^2 - 2 < 0$

$\therefore r^2 - 2 < 0$, i.e., $r^2 < 2$

We see that $r \in S$, and r is greater than u, i.e., the lub (S), which is a contradiction.

$\therefore u^2 < 2$ is not possible

Case 2. If possible, let $u^2 = 2$.

This shows that there exists a rational number whose square is 2, which is a contradiction.

$\therefore u^2$ cannot be 2

Case 3. Let $u^2 > 2$

We consider the rational number r defined in case 1.

$r > 0$, and $r - u = \frac{2(2-u^2)}{3+2u}$, from case 1

$\therefore u^2 > 2, \therefore 2 < u^2$, i.e., $2 - u^2 < 0$

$\therefore r - u < 0$, i.e., $r < u$

Again $r^2 - 2 = \frac{2-u^2}{(3+2u)^2}$, from case1

$$\therefore 2 - r^2 = = \frac{u^2 - 2}{(3+2u)^2} < 0$$

i.e., $2 - r^2 < 0$, i.e., $2 < r^2$, i.e., $r^2 > 2$

This shows that r is an upper bound which does not belong to S. Again r is less than u, i.e., the lub (S), which is a contradiction.

$\therefore u^2 > 2$ is not possible.

Thus, none of the possibilities holds good.

So, our supposition is wrong.

$\therefore$ u, i.e., lub (S) does not belong to $\underline{\underline{Q}}$.

Note : $\underline{\underline{R}}$ is a complete ordered field but $\underline{\underline{Q}}$ is not order-complete.

Do You Know?

1. If S is a nonempty set bounded above, the least upper bound of S may or may not belong to S.
2. If S is a nonempty set bounded below, the greatest lower bound of S may or may not belong to S.
3. If S is a nonempty finite set, the least upper bound and the greatest lower bound of S both exist and belong to S.

Theorem 1.8

If S is a nonempty bounded subset of $\underline{\underline{R}}$,

Then glb (S) $\leq$ lub (S)

Proof : $\because$ S is a nonempty bounded subset of $\underline{\underline{R}}$, it has least upper bound and greatest lower bound in $\underline{\underline{R}}$ (by completeness axiom)

Let a $\in$ S

$\therefore$ by definition glb (S) $\leq$ a ... (1)

and a $\leq$ lub (S) ... (2)

From (1) and (2)

glb (S) $\leq$ lub (S)

Theorem 1.9:

If A, and B are two nonempty subsets of $\underline{\underline{R}}$ with the property that $a \leq b \forall a \in A$, b $\in$ B, then

lub (A) $\leq$ glb (B)

Proof : Let b $\in$ B be arbitrary

Now $a \leq b, \forall a \in A$

$\therefore$ b is an upper bound of A, and so, by completeness axiom, it has the lub in $\underline{\underline{R}}$

Again let a $\in$A be arbitrary

Now $a \leq b, \forall b \in B$

$\therefore$ a is a lower bound of B. and so, it has the glb in $\underline{\underline{R}}$

Again a $\leq$ glb (B), $\forall a \in A$

$\therefore$ glb (B) is an upper of A

$\therefore$ lub (A) $\leq$ glb (B)

Theorem 1.10:

Let A, B be nonempty subsets of $\underline{\underline{R}}$ such that A is a subset of B, and

B is bounded above. Then A is bounded above, and

lub (A) $\leq$ lub (B)

Proof : $\because$ B is bounded above, by completeness axiom, the least upper bound of B exists in $\underline{\underline{R}}$

Let $x \in A$

$\therefore A \subseteq B$, $x \in A \Rightarrow x \in B$

$\therefore x \leq$ lub (B).

$\therefore$ A is bounded above by lub (B)

$\therefore$ by completeness axiom, A has the least upper bound in $\underline{\underline{R}}$

$\therefore$ lub (A) $\leq$ lub (B)

Do You Know?

The greatest lower bound or the least upper bound of a set may or may not belong to the set. If the least upper bound of a set belong to the set, it is called the greatest member of the set. Similarly, if the greatest lower bound of a set belongs to the set, it is called the least member of the set.

Theorem 1.11: (A) Let S be a nonempty subset of $\underline{\underline{R}}$ bounded above. Then a real number u is the least upper bound of S if and only if for any $\varepsilon > 0$, there exists $x \in S$ such that

$$u - \varepsilon < x \leq u$$

(B) Let S be a nonempty subset of $\underline{\underline{R}}$ bounded below. Then a real number w is the greatest lower bound of S if and only if for only $\varepsilon > 0$, there exists $x \in S$ such that

$$w \leq x < w + \varepsilon$$

Proof: (A) Let u = lub (S)

$\therefore x \leq u, \forall x \in S$ ... (1)

Let $\varepsilon > 0$ then $u\ \varepsilon < u$ and, therefore, $u - \varepsilon$ cannot be an upper bound of S.

$\therefore$ There exists some $x \in S$ such that $x > \varepsilon - u$ ... (2)

Conversely, let u be any real number satisfying the condirion.

$$u - \varepsilon < x \leq u$$

We are to prove that u = lub (S)

Now $x \leq u$, $\forall x \in S$, (from the given condition)

$\therefore$ u is an upper bound of S.

Let u' be any real number less than u.

$\therefore u - u' > 0$

Let $\varepsilon = u - u' > 0$

$\therefore$ There exists some $x \in S$ such that $x > u - \varepsilon$, (from the given condition)

i.e., $x > u - (u - u')$

i.e., $x > u'$, for same $x \in S$

$\therefore$ u' cannot be an upper bound of S

Thus, we see that no real number, less than u, cannot be an upper bound of S.

$\therefore$ u is the least upper bound of S.

(B) Try yourself.



CHECK YOUR PROGRESS 2

Q.1. Prove that the set of natural numbers $\underline{\underline{N}}$ is not bounded above.

Q. 2. Let A and B be two nonempty subsets of $\underline{\underline{R}}$, and let

C = {x + y : x $\in$ A and y $\in$ B}

If each of A and B has a supermum prove that C has a supremum and sup C = sup A + sup B

Q.3. Let A and B be two nonempty sets such that A, B are bounded above. Prove that AUB is bounded above, and that

sup (AUB) = max (sup (A), sup (B))

Q. 4. Let A and B be two nonempty sets such that A, B are bounded below, and $A \cap B$ is nonempty. Prove that $A \cap B$ is bounded below and max (inf (A), inf (B)) $\leq$ inf $(A \cap B)$

Q.5. Let A be a nonempty subset of $\underline{\underline{R}}$ such that A is bounded below. Let $c \in \underline{\underline{R}}$.

Define the set

$c \in \underline{\underline{R}}$ = $\{c + x : x \in A\}$

Prove that c +A is bounded below and

$\text{glb}(c + A) = c + \text{glb}(A)$

1.7 ARCHIMEDEAN PROPERTY OF REAL NUMBERS

If x, y are two positive real numbers, then there exists a natural number n such that ny > x

Proof : If possible, let there exist no natural number n for which ny > x

Then for every natural number m, my $\leq$ x.

Let us consider S defined by S = {my : $y \in \underline{\underline{N}}$ }

Clearly S is nonempty, and bounded above. x is an upper bound.

$\therefore$ by completeness axiom. S has the least upper bound. Let if be u, i.e., lub (S) = u

$\therefore$ my $\leq$ u, for all $m \in \underline{\underline{N}}$

$\because$ y > 0, u – y < u, and, therefore, u – y cannot be an upper bound of S.

$\therefore$ there exists a natural number p such that

$u - y < py \leq u$

This $\Rightarrow u < y + py$

$\Rightarrow (p+1)y > u$ ..... (1)

Now $p \in \underline{\underline{N}} \Rightarrow p+1 \in \underline{\underline{N}} \Rightarrow (p+1)y \in S$

From (1), we see that u is not the least upper bound of S, which is a contradiction. So, our supposition is wrong.

$\therefore$ ny > x

Note : The above property was named after the ancient Greek mathematician Archimedes by the Austrian mathematician otto stolz (1842-1905). The Archimedean property appears in Elements of Euclid (Definition 4, Book V). It states that, 'Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another.'

$\underline{\underline{R}}$ and $\underline{\underline{Q}}$ are Archimedean ordered field. $\underline{\underline{R}}$ is complete, but $\underline{\underline{Q}}$ is not complete.

Corollaries

1. If x is a real number, there exists a natural number n such that n > x

Proof : Let x > 0

By Archimedean property, we have ny > x, where x, y are positive real numbers, and n is a natural number. Taking y = 1, we get n > x.

Let $x \leq 0$

We take n = 1

2. If x is a positive real number, there exists a natural number n such that $0 < \frac{1}{n} < x$

Proof : By Archimedean property, we know that if x, y are positive real numbers, there exists a natural number n such that

$nx > y$

We take y = 1 $\therefore nx > 1$, i.e., $x > \frac{1}{n}$, i.e., $\frac{1}{n} < x$

Again $n > 0 \Rightarrow \frac{1}{n} > 0$, i.e., $0 < \frac{1}{n}$.

Thus, $0 < \frac{1}{n} < x$

Do You Know?

The well-ordering principle states that every nonempty set of positive integers contains a least element.

3. If x is apositive real number, there exists a natural number k such that $k - 1 \leq x < k$

Proof : By Archimedeam property, we know that if x, y are positive real numbers, there exists a natural number n such that ny > x

We take y = 1 $\therefore$ n > x

Let us consider the set

$E = \{m \in \underline{\underline{N}} : m > x\}$

Since n > x, $\therefore n \in E$

$\therefore$ E is nonempty.

$\therefore$ by well-ordering principle E contains a least element.

Let it be k.

$\because k \in E$, $\therefore$ k > x

Again k is the least element in E.

$\therefore$ k – 1 $\geq$ x

$\therefore$ k – 1 $\leq$ x < k

4. If x is a real number, there exists an integer k such that k – 1 $\leq$ x < k

Proof : (i) Let x > 0

Then the result is trun from Cor 3

(ii) Let x = 0

We take k = 1

(iii) Let $x < 0$

We assume that x is not a negative integer

Now $-x > 0$

$\therefore$ from (i), we get that there exists a natural number in such that

$m - 1 < -x < m$

$-x < m$

$\Rightarrow x > -m$

$\Rightarrow -m < x$ and $m - 1 < -x \Rightarrow x < -m + 1$

$\therefore -m < x < -m + 1$

Let $k = -m + 1$

$\therefore$ m is a natural number, k is an integer less than or equal to zero.

$\therefore$ We have $k - 1 < x < k$

If x is a negative inyeger, then $x = k - 1$

Thus $k - 1 \leq n < k$

1.8 DENSITY PROPERTY OF REAL NUMBERS

1.8.1 Density of the Rationals

If x, y are real numbers with $x < y$, there exists a rational number r such that $x < r < y$.

Proof : We have $x < y$, i.e., $y > x$,

$\therefore y - x > 0$

$\therefore$ by Archimedean property of real numbers (Cor 2), there exists a natural number n such that

$$0 < \frac{1}{n} < y - x$$

or, 0 < <ny - nx

or, $ny - nx > 1$

or, $nx + 1 < ny$ ... (1)

Now $nx \in \underline{\underline{R}}$

$\therefore$ by Archimedean property of real numbers (Cor 4), there exists an integer m such that

$m - 1 \leq nx < m$ ... (2)

Now $m - 1 \leq nx \Rightarrow m \leq nx + 1$ ... (2)

$\therefore$ from (1) and (2)

$m \leq nx + 1 < ny$ ..... (3)

Again $nx < m$

$\therefore$ $nx < m$, ny from ..... (3)

or, $x < \frac{m}{n} < y$

Here m is an integer, and n is a natural number.

$\therefore$ $\frac{m}{n}$ is a rational number

Taking $\frac{m}{n} = r$

We get $x < r < y$.

1.8.2 Density of the Irrationals

If x, y are real numbers with $x < y$, there exsits and irrational number t such that $x < t < y$.

Proof : We have

$x < y$

$\therefore \sqrt{2}x < \sqrt{2}y$

Now $\sqrt{2}x$, $\sqrt{2}y$ are real numbers.

$\therefore$ by 1. 8.1, there exists a rational number r such that

$\sqrt{2}x < r < \sqrt{2}y$

or $x < \frac{r}{\sqrt{2}} < y$

Clearly $\frac{r}{\sqrt{2}}$ is irratioral

Taking $\frac{r}{\sqrt{2}} = t$, we get

$x < t < y$

1.9 EXTENDED SET OF REAL NUMERS

We know the two symbols, viz, $+\infty$ or ∞ (plus infinity) and $-\infty$ (minius infinity). These are not real numbers. So, any result stated for real

numbers cannot be applied to ∞ and $-\infty$.

The set of real numbers $\underline{\underline{R}}$ can be extended by adjoining the two symbols by $\underline{\underline{R}}_{\infty}$, Thus

$$\underline{\underline{R}}_{\infty} = \underline{\underline{R}} \cup \{-\alpha, +\alpha\}$$

In $\underline{\underline{R}}_{\infty}$, we define

(i) for all real x, $x + \infty = \infty + x = \infty$

$x(-\infty) = (-\infty) + x = -\infty$

(ii) for $x > 0$, $x.\infty = \infty.x = \infty$

$x.(-\infty) = (-\infty).x = \infty$

(iii) for $x < 0$, $x.\infty = \infty x = -\infty$

$x(-\infty) = (-\infty).x = \infty$

(iv) $\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty$

$\infty.\infty = \infty$

$(-\infty).\infty = \infty.(-\infty) = -\infty$

The order relation in $\underline{\underline{R}}_{\infty}$ satisfies the inequality $-\infty < x < \infty, \forall x \in \underline{\underline{R}}$

If $x > 0$, $0 < x < \infty$

If $x < 0$, $-\infty < x < 0$

Let a, b, be two real numbers. Then

$[a, \infty[= \{x \in \underline{\underline{R}} : x \geq a\}$

$]a, \infty[= \{x \in \underline{\underline{R}} : x \geq a\}$

$]-\infty, b] = \{x \in \underline{\underline{R}} : x \leq b\}$

$]-\infty, b[= \{x \in \underline{\underline{R}} : x < b\}$

$]-\infty, \infty[= \{x \in \underline{\underline{R}} : -\infty < x < \infty\}$

Note : $\infty + (-\infty)$, $(-\infty) + \infty$, $0.\infty$, $\infty.0$

$0.(-\infty), (-\infty).0, \frac{\infty}{\infty}$ are not defined



1.10 LET US SUM UP

- A nonempty set equipped with one or more binary operations defined on it, is called an algebraic structure. A field is an algebraic structure.
- The set R of real numbers is a field under ordinary addition and multiplication.

- The field of real numbers is an ordered field. The field of rational numbers is also an order field. But the field of complex numbers is not ordered.
- Every nonempty subset of $\underline{\underline{R}}$ that is bounded above has the least upper bound (supermom) in $\underline{\underline{R}}$. This is known as the completeness axiom.
- $\underline{\underline{R}}$ is a complete ordered field.
- If x, y are two positive real numbs, then there exists a natural known as n such that ny > x. This is known as tho Archimedean property of real numbers.
- $\underline{\underline{R}}$ is an Archimedean ordered field.
- If x, y are real numbers with x < y, there exists a rational number r such that x < r < y.
- If x, y are real numbers with x < y, there exists an irrational number t such that x < t<y.



1.11 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progresss 1

Ans to Q No 1:

(i) $|x + y + z|$

$= |(x + y) + z|$

$\leq |x + y| + |z|$

$\leq |x| + |y| \mid |z|$

(ii) $|xyz|$

$= |(xy)z|$

$= |xy| \, |z|$

$= |x| \, |y| \, |z|$

Ans to Q No 2: Try yourself

Ans to Q No 3: Try yourself

Ans to Q No 4: Try yourself

Ans to Q No 5: $|x - y = |x + (-y)|$

$\leq |x| + |y|$

$= |x| + |y|$

i.e., $|x - y| \leq |x| + |y|$

Ans to Q No 6: $a < x < b$

iff $a - \frac{a+b}{2} < x - \frac{a+b}{2} < b - \frac{a+b}{2}$

iff $\frac{2a-a-b}{2} < x - \frac{a+b}{2} < \frac{-b-a-b}{2}$

iff $\frac{a-b}{2} < x - \frac{a+b}{2} < \frac{b-a}{2}$

iff $-\frac{b-a}{2} < x - \frac{a+b}{2} < \frac{b-a}{2}$

iff $\left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$

Do you Know

The meaning of "if and only if" (shortened as iff) is as follows.
'The truth of either one of the connected statements required the truth of the other.' Logical symbol for "iff" is $\Leftrightarrow$.

Note: In the above example it is shown that if $a < x < b$, then $\left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$,

and conversely if $\left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$, then $a < x < b$.

Ans to Q No 7: We have

$$\max\{a, b\} = \begin{cases} a, \text{ if } a > b \\ b, \text{ if } a < b \\ \frac{a+b}{2}, \text{ if } a = b \end{cases}$$

$$\min\{a,b\} = \begin{cases} b, \text{ if } a > b \\ a, \text{ if } a < b \\ \frac{a+b}{2}, \text{ if } a = b \end{cases}$$

Therefor

if a > b, max {a, b} + min {a, b} = a + b

if a < b, max {a, b} + min {a, b) = b + a

if a = b, max {a, b} + min {a, b} = a + b

Agin

if a > b, max {a, b} – min {a, b} = a – b = {a – b|

if a < b, max {a, b} – min {a, b) = b – a = |a – b|

if a = b, max {a, b} – min {a, b} = 0 = |a – b|

Thus, for all real numbers a, and b

max {a, b} + min {a, b} = a + b ... (1)

max {a, b} – min {a, b} = |a – b| ... (2)

Adding (1) and (2), we get

2 max {a, b} = a + b + |a – b|

i.e., max {a, b} = $\frac{1}{2}$ [a + b + |a – b|]

Again subtracting (2) from (1), we get

2 min {a, b} = a + b – |a – b|

i.e., min {a, b} = $\frac{1}{2}$ [a + b – |a – b|]

Ans to Q No 8:

$\left|\frac{x+5}{3x-1}\right| = 4$, (given)

$\therefore \frac{x+5}{3x-1} = 4$ or $\therefore \frac{x+5}{3x-1} = -4$

Now $\frac{x+5}{3x-1} = 4$

$\Rightarrow$ x + 5 = 4 (3x – 1)

$\Rightarrow$ x + 5 = 12x – 4

$\Rightarrow$ 12x – x = 5 + 4

$\Rightarrow$ 11x = 9

$\Rightarrow x = \frac{9}{11}$

Again, $\frac{x+5}{3x-1} = -4$

$\Rightarrow x + 5 = -4(3x - 1)$

$\Rightarrow x + 5 = -12x + 4$

$\Rightarrow x + 12x = 4 - 5$

$\Rightarrow 13x = -1$

$\Rightarrow x = -\frac{1}{13}$

Thus, $x = \frac{9}{11}, -\frac{1}{3}$

Ans to Q No 9: We have

$|a + b| = |a| + |b|$

iff $|a + b|^2 = (|a| + |b|)^2$

iff $(a + b)^2 = |a|^2 + |b|^2 + 2|a|\,|b|$

iff $a^2 + b^2 + 2ab = a^2 + b^2 + 2|ab|$

iff $ab = |ab|$

iff $ab \geq 0$

Ans to Q No 10: We have

$|a + b| < |a| + |b|$

iff $|a + b|^2 < (|a| + |b|)^2$

iff $(a + b)^2 < |a|^2 + |b|^2 + 2|a|\,|b|$

iff $a^2 + b^2 + 2ab < a^2 + b^2 + 2|ab|$

iff $ab < |ab|$

iff $ab < 0$

Ans to Q No 11: (i) We have

$|x - 5| < 2$

$\Leftrightarrow \quad 5 - 2 < x < 5 + 2$

$\Leftrightarrow \quad 3 < x < 7$

$\therefore$ the statement is true.

(ii) $|x - 4| < 1$

$\Leftrightarrow \quad 4 - 1 < x < 4 + 1$

$\Leftrightarrow \quad 3 < x < 5$

$\therefore$ The statement is not true.

(iii) $|x - 1| < 2$

$\Leftrightarrow \quad 2 - 1 < x < 2 + 1$

$\Leftrightarrow$ 1 < x < 3

$\therefore$ the statement is true.

Ans to Q No 12: We know that

$$|x| = \begin{cases} x, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -x, \text{if } x < 0 \end{cases}$$

Here $a \neq 0$

Let a > 0

$$\therefore f(a) = \frac{|a|}{a} = \frac{a}{a} = 1$$

$$f(-a) = \frac{|-a|}{-a} = \frac{a}{-a} = -1$$

$\therefore |f(a) - f(-a)| = |1 - (-1)| = |1 + 1| = |2| = 2$

Let a < 0

$$\therefore f(a) = \frac{|a|}{a} = \frac{-a}{a} = -1$$

$$f(-a) = \frac{|-a|}{-a} = \frac{-a}{-a} = 1$$

$\therefore |f(a) - f(-a)| = |-1 - 1| = |-2| = 2$

Ans to Q No 13: We have

$|a + b| \leq |a| + |b|$

$\Rightarrow 1 + |a + b| \leq 1 + |a| + |b|$

$$\Rightarrow \frac{1}{1+|a+b|} \geq \frac{1}{1+|a|+|b|}$$

$$\Rightarrow -\frac{1}{1+|a+b|} \leq -\frac{1}{1+|a|+|b|}$$

$$\Rightarrow 1-\frac{1}{1+|a+b|} \leq 1-\frac{1}{1+|a|+|b|}$$

$$\Rightarrow \frac{1+|a+b|-1}{1+|a+b|} \leq \frac{1+|a|+|b|-1}{1+|a|+|b|}$$

$$\Rightarrow \frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|} \text{ ..... (1)}$$

again $1 + |a| + |b| \geq t\,|a|$

$$\Rightarrow \quad \frac{1}{1+|a|+|b|} \le \frac{1}{1+|a|}$$

$$\Rightarrow \quad \frac{|a|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} \qquad \text{... (2)}$$

Similarly $\frac{|b|}{1+|a|+|b|} \le \frac{|b|}{1+|b|}$

Again (2) and (3), we get

$$\frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

$$\Rightarrow \quad \frac{|a|+|b|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \qquad \text{... (4)}$$

From (1) and (4), we get

$$\frac{|a+b|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

Ans to Q No 14: We have

$|a - b| \le |a| + |b|$

$$\Rightarrow \quad 1 + |a - b| \le 1 + |a| + |b|$$

$$\Rightarrow \quad \frac{1}{1+|a-b|} \ge \frac{1}{1+|a|+|b|}$$

$$\Rightarrow \quad -\frac{1}{1+|a-b|} \le -\frac{1}{1+|a|+|b|}$$

$$\Rightarrow \quad 1-\frac{1}{1+|a-b|} \le 1-\frac{1}{1+|a|+|b|}$$

$$\Rightarrow \quad \frac{1+|a-b|-1}{1+|a-b|} \le \frac{1+|a|+|b|-1}{1+|a|+|b|}$$

$$\Rightarrow \quad \frac{|a-b|}{1+|a-b|} \le \frac{|a|+|b|}{1+|a|+|b|} \qquad \text{... (1)}$$

Again $1 + |a| + |b| \ge a + |a|$

$$\Rightarrow \quad \frac{1}{1+|a|+|b|} \le \frac{1}{1+|a|}$$

$$\Rightarrow \quad \frac{|a|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} \qquad \text{... (2)}$$

Similarly $\frac{|b|}{1+|a|+|b|} \le \frac{|b|}{1+|b|}$ ... (3)

Again (2) and (3), we get

$$\frac{|a|}{1+|a|+|b|}+\frac{|b|}{1+|a|+|b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$$

$$\Rightarrow \quad \frac{|a|+|b|}{1+|a|+|b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|} \qquad \text{... (4)}$$

$$\Rightarrow \quad \frac{|a|+|b|}{1+|a-b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$$

From (1) and (4), We get

$$\Rightarrow \quad \frac{|a-b|}{1+|a-b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$$

Check Your Progresss 2

Ans to Q No 1: If possible, Let $\underline{\underline{N}}$ be bounded above. Therefore, by completeness axiom, it has the least upper bound. Let it be u.

$\therefore$ u – 1 is not an upper bound for $\underline{\underline{N}}$.

$\therefore$ There exists a $\in \underline{\underline{N}}$ such that u – 1< a.

i.e., u < 1 + a

But a $\in \underline{\underline{N}} \Rightarrow$ 1 + a $\in \underline{\underline{N}}$

$\therefore$ u is not an upper bound for $\underline{\underline{N}}$, which is a contradiction.

$\therefore$ Our supposition is wrong.

$\therefore$ $\underline{\underline{N}}$ is not bounded above.

Ans to Q No 2: Let u = sup (A)

w = sup (B)

$c = \{x+2 : x \in A \text{ and } y \in B\}$

It is required to prove that u + w = sup (c)

Let z $\in \subset$ be arbitray

Then there exist x $\in$ A and y $\in$ B

Such that

z = x + y

Now x $\le$ u, and y $\le$ w

$\therefore$ x + y $\le$ u + w

$\Rightarrow$ z $\le$ u + w

$\therefore$ u + w is an upper bound for c.

Let a be any real number such that a < u + w

$\therefore$ a – w < u i. e., a – q < sup (A)

$\therefore$ a – w is not an upper bound for A,

$\therefore$ There exists some x $\in$ A such that a – w < x

or a – x < w, i.e., a – x < sup (B)

$\therefore$ a – x is not an upper bound for B.

$\therefore$ There exists some y $\in$ B such that a – x < y

i.e., a < x + y, where x + y $\in$ c.

$\therefore$ a is not an upper bound for c.

$\therefore$ u + w is the least upper bound for c.

Thus, sup (c) = u + w

= sup (A) + sup (B)

Ans to Q No 3: Let u = max (sup (A), sup (B)

Clearly sup (A) $\leq$ u,

sup (B) $\leq$ u

Let $x \in A \cup B$

$\Rightarrow$ x $\in$ A or x $\in$ B (or both)

If x $\in$ A, then x $\leq$ sup (A) $\leq$ u

If x $\in$ B, then x $\leq$ sup (B) $\leq$ u

$\therefore$ $A \cup B$ is bound above, and u is an upper bound fo $A \cup B$

$\therefore$ by completeness axiom, $A \cup B$ has the least upper bound, i.e., sup ($A \cup B$) exists.

Thus, sup ($A \cup B$) $\leq$ u

i.e., sup ($A \cup B$) $\leq$ max (sup (A), sup (B)) ... (1)

We know that $A \subseteq A \cup B$

$\therefore$ sup (A) $\leq$ sup ($A \cup B$), [Theorem1, 10]

Also $B \subseteq A \cup B$

$\therefore$ sup (B) $\leq$ sup ($A \cup B$), [Theorem 1.10]

$\therefore$ max (sup(A), sup (B)) $\leq$ sub ($A \cup B$) ... (2)

From (1) and (2), sup ($A \cup B$) = max (sup (A), sup (B))

Ans to Q No 4: A, B are bounded below

$\therefore$ inf (A), inf (B) exist.

Moreover $A \cap B \neq \phi$

Let x $\in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$x \in A \Rightarrow \inf(A) \leq x$

$x \in B \Rightarrow \inf(B) \leq x$

$\therefore$ inf (A), and inf (B) are both lower bounds for $A \cap B$

$\therefore$ $\inf(A) \leq \inf A \cap B$ ... (1)

$\inf(B) \leq \inf A \cap B$ ..... (2)

From (1) and (2)

max (inf (A), inf (B) $\leq \inf(A \cap B)$

Ans to Q No 5: Let $x \in A$

$\therefore$ glb (A) $\leq x, \forall x \in A$

$\therefore$ $c + glb(A) \leq c + x$

Now $c + x \; c \in +A$

$\therefore$ c + glb (A) is a lower bound for c + A

$\therefore$ $c + glb(A) \leq glb(c + A)$ ..... (1)

Now glb (c + A) $\leq z, \forall z \in c + A$

$\Rightarrow glb(c + A) \leq c + x,$ where $z = c + x, x \in A$

$\Rightarrow glb\ (c + A) - c \leq x, \forall x \in A$

or glb (c + A) $\leq$ c + glb (A) ..... (2)

From (1) and (3)

glb (c + A) = c + glb (A)



1.12 FURTHER READING

1) Principles of Mathematical Analysis by Walter Rudin

2) Methods of real Analysis by Richand R. Goldbery

3) Mathematical Analysis by S. C. Malik and Savita Arora

4) A Course of Mathematical Analysis by shanti Narayan.



1.13 MODEL QUESTIONS

Q 1: If a, b are real numbers and $\varepsilon > 0$, show that $|a - b| < \varepsilon$ if and only if $b - \varepsilon < a < b + \varepsilon$.

Q 2: If a, b are real numbers, show that

(i) $\sqrt{a^2 + b^2} \leq |a| + |b|$

(ii) $\sqrt{|a + b|} \leq \sqrt{|a|} + \sqrt{|b|}$

Q 3: Let $S = \{x \in \underline{\underline{Q}} : x > 0 \text{ and } x^2 < 3\}$

Show that S is nonempty and bounded above, but it does not have the supremum in $\underline{\underline{Q}}$.

Q 4: Give an example of an ordered field which is not complete. Justify your answer.

Q 5: Find the suprema and infima, if they exist, of the following sets

(i) $\left\{\frac{n-1}{n} : n \in \underline{\underline{N}}\right\}$

(ii) $\left\{\frac{n}{n+1} : n \in \underline{\underline{N}}\right\}$

(iii) $\left\{\frac{1}{n} : n \in \underline{\underline{Z}}, n \neq 0\right\}$

Q 6: Show that every finite subset of the set of real numbers is bounded.

Q 7: Show that every subset of a bounded set is bounded.

Q 8: Let A be a nonempty subset or $\underline{\underline{R}}$ such that A is bounded above.

Let $c \in \underline{\underline{R}}$. Define the set

$cA = \{cx : x \in A\}$

Prove that if $c > 0$, then cA is bounded above and sup (cA) = c sup (A)

Q 9: For any set A, define the set $-A = \{y \in \underline{\underline{R}} : \exists\, x \in A$ such that $y = -x\}$

Prove that if A is nonempty and bounded, then sup (–A) = – inf (A)

Q 10: Prove that between any two real numbers there is an irrational number.

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UNIT 2 : REAL NUMBER SYSTEM-II

UNIT STRUCTURE

2.1 Learning Objectives

2.2 Introduction

2.3 Neighbourhood of a Point

2.4 Limit Point of a Set

2.5 Bolzano-Weierstrass Theorem

2.6 Interior Point, Interior

2.7 Open Set

2.8 Closed Set

2.9 Closure of a Set

2.10 Countability of Sets

2.11 Let Us Sum Up

2.12 Answers to Check Your Progress

2.13 Further Readings

2.14 Model Questions

2.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- neighbourhood of a point on the real line.
- limit point of a set, interior of a set.
- bolzano-Weierstrass Theorem
- open sets, closed sets, closure of a set
- countability of sets.

2.2 INTRODUCTION

In this unit we will discuss the points and subsets of the real line such as neighbourhood of a point, open sets, closed sets, limit points of a set, etc. These are known as the topological properties of the real line.

2.3 NEIGHBOURHOOD OF A POINT

Let p be a real number. A subset S of $\underline{\underline{\mathrm{R}}}$ is said to be a neighbourhood of p if there exists an open interval]a, b[such that

$$p \in]a,\ b[\subseteq \mathrm{S}$$

This is equivalent to saying that there exists an open interval of the form $]p-\delta,\ p+\delta[$ for same $\delta > 0$ such that

$$]p-\delta, p+\delta[\subseteq s$$

$]p-\delta, p+\delta[$ is called the δ-neighbourhood of p.

For convenience, we will write nbd for neighbourhood.

Note: 1. An open bounded interval containing the point p is a neighbourhood of p.

2. A closed bounded interval containing the point p may not be a neighbourhood of p.

Let us consider the closed interval [0, 1]. [0, 1] is not a nbd of 9, for there does not exist any open interval containing 0 which is a subset of [0, 1]. Similarly, [0, 1] is not a nbd of 1.

A closed bounded interval [*a, b*] is a nbd of each of the points of] *a, b* [, but not of the end points *a* and *b*.

3. The null set ϕ is a nbd of each of its point in the sense that there is no point in ϕ of which it is not a nbd.

4. The set of real numbers R is a nbd of each real number p, for any open interval containing p is a subset of R.

5. The set of rational numbers Q is not a nbd of any of its points p, for any open interval containing p will contain infinite number of irrotional numbers. Clearly the open interval cannot be a subset of Q.

6. The singleton set {p} for any $\mathrm{p} \in \mathrm{R}$ is not a nbd of p, for there does not exist any open interval containing p which is a subset of {p}.

7. A nonempty finite subset S of R is not a nbd of any point.

It S is a nbd of p, then there exists an open interval containing p which is subset of S. Now the open interval contains an infinite number of points, whereas S is finite. Thus, S cannot be a nbd of p.

Theorem 2.1 The union of any two nbds of a point is a nbd of the point.

Proof : Let S_1, S_2 be two subsets of R.

Let $p \in R$

Let S_1, S_2 be two nbds of p.

$\because$ S_1 is a nbd of p, there exists open interval [a_1, b_1 [such that

$p \in] a_1, b_1 [\subseteq S_1$

Again, $\because$ S_2 is a nbd of p, there exists open interval [a_2, b_2 [such that

$p \in] a_2, b_2 [\subseteq S_2$

Now $p \in] a_1, b_1 [\Rightarrow a_1 < p < b_1$

$p \in] a_2, b_2 [\Rightarrow a_2 < p < b_2$

Now $a_1 < b_1, a_1 < b_2, a_2 < b_1, a_2 < b_2$

Let a_3 = min $\{a_1, a_2\}$

b_3 = max $\{b_1, b_2\}$

Then $]a_1, b_1[\cup]a_2, b_2[=]a_3, b_3[$

Also $p \in]a_3, b_3[$

Now $]a_1, b_1[\leq S_1 \cup S_2$, $]a_2, b_2[\subseteq S_1 \cup S_2$

$\therefore]a_3, b_3[=]a_1, b_1[\cup]a_2, b_2[\subseteq S_1 \cup S_2$

Thus $p \in]a_3, b_3[\subseteq S_1 \cup S_2$

$\therefore$ $S_1 \cup S_2$ is a nbd of p.

Note: The union of a finite number of nbds of p is a nbd of p.

Theorem 2.2 The intersection of any two nbds of a point is a nbd of the point.

Proof : Let S_1, S_2 be two subsets of R.

Let $p \in R$

Let S_1, S_2 be two nbds of p.

$\because$ S_1 is a nbd of p, there exists open interval] a_1, b_1 [such that

$p \in]a_1, b_1[\subseteq S_1$

Again, $\because$ S_2 is a nbd of p, there exists open interval] a_2, b_2 [such that

$p \in]a_2, b_2[\subseteq S_2$

Now $a_1 < b_1, a_1 < b_2, a_2 < b_1, a_2 < b_2$

Let a_3 = max $\{a_1, a_2\}$

b_3 = min $\{b_1, b_2\}$

Then $]a_1, b_1[\cap]a_2, b_2[=]a_3, b_3[$

Also $p \in]a_3, b_3[$

Now $]a_3, b_3[=]a_1, b_1[\cap]a_2, b_2[\subseteq]a_1, b_1[\leq S_1$

$]a_3, b_3[=]a_1, b_1[\cap]a_2, b_2[\subseteq]a_2, b_2[\subseteq S_2$

$\therefore]a_3, b_3[\leq S_1 \cap S_2$

This $p \in]a_3, b_3[\leq S_1 \cap S_2$

$\therefore S_1 \cap S_2$ is a nbd of p.

Note 1. The intersection of a finite number of nbds of p is a nbd of p.

2. The intersection of an infinite number of nbds of p may not be a nbd of p.

Let us consider the family of open intervals $]-1,1[, \left]-\frac{1}{2}, \frac{1}{2}\right[, \left]-\frac{1}{3}, \frac{1}{3}\right[, \left]-\frac{1}{4}, \frac{1}{4}\right[, ..., \left]-\frac{1}{n}, \frac{1}{n}\right[, ...$

Each of these open intervals is a nbd of 0.

Now $]-1,1[\cap \left]-\frac{1}{2}, \frac{1}{2}\right[\cap \left]-\frac{1}{3}, \frac{1}{3}\right[\cap ... \cap \left]-\frac{1}{n}, \frac{1}{n}\right[\cap = \bigcap_{n=1}^{\infty} \left]-\frac{1}{n}, \frac{1}{n}\right[= \{0\}$

Clearly {0} is not a nbd of 0.



CHECK YOUR PROGRESS-1

Q.1: If S is a nbd of a point p, and $S \subseteq T$, show that T is also a nbd of p.

Q.2: Can a set S be a nbd of any point of R – S? Justify your answer.

Q.3: Give an example of each of the following :

(a) a set which is a nbd of each of its points.

(b) a set which is not ao nbd of any of its points.

Do You Know ?

Let p be any point on the real line. Let $\delta > 0$ be a real number. A set consisting of all those points on the real line which are at a distance δ from p is called a neighbourhood (nbd) of p. Thus, the set $\{x : x \in R, |x - p| < \delta$

i.e., $\{x : p - \delta < x < p + \delta\}$

i.e., $]p - \delta, p + \delta[$

is a nbd of p.

2.4 LIMIT POINT OF A SET

Let S be a subset of R. A real number p is said to be a limit point (or a cluster point, or an accumulation point) of S if every nbd of p contains a point of S other than p.

The null set has no limit point.

Note : The set of all limit points of S is called the derived set of S. Generally it is denoted by S'.

Examples

1. Let us consider the set S = [0, 1 [

Clearly $0 \in S$

Let us consider the nbd] – 0.5, 0.5 [of 0.

We have 0.1 $\in$] – 0.5, 0.5 [

Also 0.1 $\in$ S.

Thus we can find the point 0.1 $\in$ S. Also we note that $0.1 \neq 0,$ i.e, the nbd] – 0.5, 0.5 [of 0 contains point of S other than 0.

In this way, we will see that every nbd of 0 contains point of S different from 0.

$\therefore$ 0 is a limit point of S.

We note that the limit point 0 belongs to S.

2. Let us consider the set

$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots\ldots\right\}$$

$$= \left\{\frac{1}{n} : n \in N\right\}$$

Here 0 does not belong to S.

Let us consider the nbd] – 0.5, 0.5 [of 0.

We have 0.25 6] – 0.5, 0.5 [

Also $0.25 = \frac{1}{4} \in S$

Thus we can find 0.25 $\in$ S.

Also we note that $0.25 \neq 0,$ i.e. the nbd] – 0.5, 0.5 [of 0 contaions point of S other than 0.

In this way, we will see that every nbd of 0 contains point of S different from 0.

$\therefore$ 0 is a limit point of 0.

Also we note that 0 is the only limit point of S.

We note that the limit point 0 does not belong to S.

3. Let us consider the set N of natural numbers.

For every real number p, we can always find $\delta > 0$ such that $]p-\delta,\ p+\delta[$ does not contain a point of N different from p.

Thus, N has no limit point.

4. Let us consider the set Q of rational numbers.

We know that between any two rational numbers, there exist infinite number of rational numbers. Also there exist infinite number of rationals between any two irrational numbers.

Thus, every point of R is a limit point of Q.

From the above examples, we note the following :

FACTS

1. A limit point of a set may or may not belong to the set.
2. A set may have no limit point.
3. A set may have only one limit point.
4. A set may have more than one limit point.

Theorem 2.3 : Let A, B be two subsets of R. Then

(a) $A \subseteq B \Rightarrow A' \subseteq B'$

(b) $(A \cup B)' = A' \cup B'$

(c) $(A \cap B)' = A' \cap B'$

Proof (a) Let $p \in A'$

This $\Rightarrow$ p is a limit point of A.

$\therefore$ For each $\delta > 0$, the nbd $]p-\delta,\ p+\delta[$ contains a point, say p_1, of A other than p.

Now $p_1 \in A \Rightarrow p_1 \in B, \because A \subseteq B$

Thus the nbd $]p-\delta,\ p+\delta[$ contains a point $p_1 \in B$ other than p.

$\therefore$ p is a limit point of B, i.e, $p \in B'$

Thus $p \in A' \Rightarrow p \in B'$

$\therefore A' \subseteq B'$

(b) We know that $A \subseteq A \cup B$

$\therefore A' \subseteq (A \cup B)'$, by (a)

Again, $B \subseteq A \cup B$

$\therefore B' \subseteq (A \cup B)'$, by (a)

$\therefore A' \cup B' \subseteq (A \cup B)'$ (1)

Let $\therefore p \in (A \cup B)'$

$\therefore$ p is a limit point of $A \cup B$.

$\therefore$ Every nbd of p contains a point $q \in A \cup B,\ q \neq p$

$\therefore$ Every nbd of p contains a point ($q \in A$ or $q \in B$), $q \neq p$

$\therefore$ Every nbd of p contains a point $q \in A,\ q \neq p$

Or every nbd of p contains a point $q \in B,\ q \neq p$

$\therefore$ p is a limit point of A or p is a limit point of B.

i.e., $p \in A'$ or $p \in B'$

i.e., $p \in A' \cup B'$

$\therefore (A \cup B)' \subseteq A' \cup B'$ (2)

From (1) and (2), we get

$(A \cup B)' = A' \cup B'$

(c) We know that $A \cap B \subseteq A$

$\therefore\ (A \cap B)' \subseteq A'$, by (a)

Again $A \cap B \subseteq B$

$\therefore\ (A \cap B)' \subseteq B'$, by (b)

$\therefore\ (A \cap B)' \subseteq A' \cap B'$

Do You Know ?

1. Every point of [*a, b*] is its limit point. A point not belonging to [*a, b*] is not a limit point. Thus $[a,b]' = [a,b]$
2. Every point of] *a, b* [is its limit point. Also the points, *a, b* are its limit point. Thus,

$$]a,b[' = [a, b].$$

Note: $(A \cap B)' \neq A' \cap B'$

Let us consider $A =]2, 3[,\ B =]3, 4['$

$A' =]2, 3[' = [2, 3]$

$B' =]3, 4[' = [3, 4]$

$\therefore A' \cap B = \{3\}$

Again $A \cap B = \phi$

$$(A \cap B)' = \phi' = \phi$$

$$\therefore (A \cap B)' \neq A' \cap B'.$$

Theorem 2.4 : Let S be a subset of $\underline{\underline{R}}$. Let p be a limit point of S. Then every nbd of p contains an infinite number of elements of S.

Proof : Let p be a limit point of S. Then every nbd of p contains at least one point of S different from p. Let $]p - \delta_1, p + \delta_1[$ be a nbd which contains a point $p_1 \in S, p_1 \neq p$

Let $|p_1 - p| = \delta_2 < \delta_1$

Now, let us consider the nbd $]p - \delta_2, p + \delta_2[$ of p.

It must contain a $p_2 \in S, p_2 \neq p$.

Continuing the above process for the nbd $]p - \delta_3, p + \delta_3[$ of p, where $\delta_3 = |p_2 - p|$, and so on, we conclude that the nbd $]p - \delta_1, p + \delta_1[$ contains an infinite number of elements of S.

Theorem 2.5 : A finite set has no limit points.

Proof: Let S be a finite set.

Let S = $\{x_1, x_2, ..., x_m\}$, where all the x_1's are distinct.

Let p be a real number.

Now p cannot be a limit point of S, for if p is a limit point of S, then every nbd of p must contain an infinite number of elements of S, which is not possible, since S is a finite set.

$\therefore$ S has no limit point.

Do You Know ?

Let S be a subset of R. A real number p is a limit point of S if every nbd of p contains an infinite number of elements of S.

2.5 BOLZANO-WEIERSTRASS THEOREM

We have seen that a finite set has no limit point. An infinite set may or may not have limit points. For example, N has no limit pointl whereas every point of Q is a limit point of Q. Now one pertinent question arises. Under what conditions, an infinite set will have a limit point? (A finite set has no limit point). The problem was first studied by the Bohemian mathematician Bernard Bolzano (5 October 1781 - 18 December 1848), and later proved

by the German mathematician Karl Theodor Wilhelm Weierstrass (31 October 1815 - 19 February 1897). We are to going to prove this theorem, known as the Bolzano-Weierstrass Theorem.

Theorem 2.6 (Bolzano-Weierstrass)

Every infinite bounded subset of R has a limit point of R.

Proof : Let A be an infinite, bounded subset of R.

Since A is bounded, it must have both lower bound and upper bound.

Let m be a lower bound, and x be an upper bound of A.

Clearly $m \leq x \leq u, \forall x \in \mathrm{A}$.

Let S be a subset of R defined by $\mathrm{S} = \{x \in \underline{\underline{\mathrm{R}}} : x$ exceeds at most finite number of elements of A}

Thus m $\in$ S, M $\notin$ S

Clearly $\mathrm{S} \neq \phi$.

Since S is nonempty and bounded above, therefore, by completeness Axiom it must have a supremum in R.

Let p be the supremum of S.

$\therefore$ There is at least one element Z $\in$ S such that $\mathrm{Z} > \mathrm{p} - \delta$ for same $\delta > 0$.

Now Z $\in$ S. Therefore, Z exceeds at the most a finite number of elements of A. Thus the number of elements of A lying on the left of $\mathrm{p} - \delta$ is finite at the most, whereas the number of elements of A lying on the right of $\mathrm{p} - \delta$ is infinite.

Since p is the supremum of S, $\mathrm{p} + \delta \notin \mathrm{S}$. Thus, $\mathrm{p} + \delta$ exceeds an infinite number of elements of A. This implies that the number of elements of A lying on the left of $\mathrm{p} + \delta$ is infinite.

Thus, the number of elements on the right of $\mathrm{p} - \delta$ is infinite, and the number of elements on the left of $\mathrm{p} + \delta$ is infinite.

Hence, $]\mathrm{p} - \delta, p + \delta[$ contains an infinite number of elements of A.

$\therefore$ P is a limit point of A.

2.6 INTERIOR POINT, INTERIOR

We recall that a subset S $\subseteq$ R is said to be a nbd of a real number p if there exists an open interval] *a, b* [such that $p \in]a, b[\subseteq$ S. A nbd of p is also denoted by N(p).

Now, we define an interior point of a subset S of R.

Definition : Let S be a subset of R. A point $p \in S$ is said to be an interior point of S if there exists a nbd N (p) of p such that N (p) $\in$ S.

The set of all interior points of S is called the **interior** of S. Generally it is denoted by int S.

Note: The null set ϕ has no interior point. int ϕ = ϕ.

Example : Let us consider

$$S = \left\{ \frac{1}{n} : n \in N \right\}$$

$$= \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\}$$

Let $p \in S$

Clearly, every nbd of p contains points not belonging to S. So p cannot be an interior point of S.

$\therefore$ int S = ϕ

Explanation : Let $p = \frac{1}{2}, \delta = 0.1.$ The nbd $]\frac{1}{2} - 0.1, \frac{1}{2} + 0.1[$ of $\frac{1}{2}$ is not a subset of S, as it contains points not belonging to S.

2.7 OPEN SET

Let S be a subset of R. S is said to be an open set if every point of S is an interior point of S.

Note: For the null set ϕ, the condition required to be an open set is vacuously satisfied. ϕ is an open set.

Example : Let us consider

$$S = \left\{ \frac{1}{n} : n \in N \right\}$$

$$= \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\}$$

We know that no point of S is an interior point of S. Therefore, S is not an open set.

Do You Know ?

"In mathematics and logic, a vacuous truth is a conditional or universal statement (a universal statement that can be converted to a conditional statement) that is true because the antecedent cannot be satisfied." — Wikipedia.



CHECK YOUR PROGRESS- 2

Q.1: What is the interior of N? Is N an open set?

Q.2: What is the interior of Z? Is Z an open set?

Q.3: What is the interior of Q? Is Q an open set?

Q.4: What is the interior of R? Is R an open set?

Q.5: Let *a, b* be real numbers, and $a < b$. Example if (i)] *a, b* [, (ii) [*a, b* [, (iii)] *a, b*], (iv) [*a, b*] are open.

Q.6: Show that a nonempty finite set is not open.

Do You Know ?

1.] *a, b* [is an open set
2. [*a, b*] is not open

a, b are real numbers, $a < b$

Theorem 2.7 : Let S be a subset of R. Then S is an open set if and only if S = int S.

Proof : For the empty set ϕ, int $\phi = \phi$. Also ϕ is an open set. We assume that $S \neq \phi$.

Let S be open

Let $p \in S$

Then p is an interior point of S, i.e, $p \in$ int S

Then, $p \in S \Rightarrow p \in$ int S

$\therefore S \subseteq \text{int } S$. (1)

Let q be an interior point of S. Then, by definition, $q \in S$

Thus, $q \in$ int $S \Rightarrow q \in S$.

$\therefore$ S = int S

Conversely, Let S = int S.

Let $p \in S$

$\Rightarrow p \in$ int S ($\because$ S = int S)

Thus, every point of S is an interior point of S.

$\therefore$ S is open.

Theorem 2. 8 : The union of two open sets in R is open.

Proof : Let G_1, G_2 be two open sets in R.

Let $p \in G_1 \cup G_2$

$\Rightarrow p \in G_1$ or $p \in G_2$

Let $p \in G_1$.

$\because$ G_1 is open, $\because$ p is an interior point of G_1.

$\therefore$ There exists a nbd N(p) of p such that $N(p) \subseteq G_1$.

Thus, $N(p) \subseteq G_i \subseteq G_1 \cup G_2$

$\therefore$ p is an interior point of $G_1 \cup G_2$

$\therefore$ $G_1 \cup G_2$ is an open set.

Similarly, if $p \in G_2$, it can be proved that $G_1 \cup G_2$ is an open set in R.

Theorem 2. 9 : The union of a finite number of open sets in R is open.

Proof : Let G_1, G_2, G_3, ..., G_n be n open sets in R.

Let $G = G_1 \cup G_2 \cup G_3 \cup .... \cup G_n$

Let $p \in G = G_1 \cup G_2 \cup G_3 ..... \cup G_n$

$\Rightarrow$ p belongs to at least one of the sets, say, G_1.

Now $p \in G_1$, and G_1 is open.

$\therefore$ p is an interior point of G_1

$\therefore$ There exists a nbd N(p) of p such that $N(p) \subseteq G_1$.

Thus, $N(p) \subseteq Gi \subseteq G = G_1 \cup G_2 \cup G_3 .... \cup G_n$

$\therefore$ p is an interior point of F.

$\therefore$ G is an open set.

Do You Know ?

An indexed class of sets, denoted by $\{G_i : i \in I\}$ assigns a set G_i to each $i \in I$. The set I is called the index set, the sets G_i are called indexed sets, and each $i \in I$ is called an index.

Theorem 2.10 : The union of an arbitrary collection of open sets in $\underline{\underline{R}}$ is open.

Proof: Let $\{G_i : i \in I\}$ be an arbitrary collection of open sets.

Let $G = \cup_{i \in I} G_i$

Let $p \in G$.

Then p belongs to at least one open set. Let it be G_j, $j \in I$

Now $p \in G_j$, and G_j is open.

$\therefore$ p is an interior of G_j

$\therefore$ There exists a nbd N(p) of p such that $N(p) \leq G_j$

Thus, $N(p) \leq G_j \leq G = \cup_{i \in I} G_i$

$\therefore$ p is an interoor of G.

$\therefore$ G is an open set.

Theorem 2.11 : The intersection of two open sets in R is open.

Proof : Let G_1, G_2 be two open sets in R. If $G_1 \cap G_2 = \phi$, there is nothing to prove. ϕ is an open set.

We assume that $G_1 \cap G_2 \neq \phi$

Let p $\in G_1 \cap G_2$

$\Rightarrow p \in G_1$ and $p \in G_2$

Now, $p \in G_1$, and G_1 is open.

$\therefore$ p is an interior point of G_1.

$\therefore$ There exists $\delta_1 > 0$ such that

$] p - \delta_1, p + \delta_1 [\subseteq G_1$

Again $p \in G_2$, and G_2 is open.

$\therefore$ p is an interior point of G_2.

$\therefore$ There exists $\delta_2 > 0$ such that

$] p - \delta_2, p + \delta_2 [\subseteq G_2$

Let δ = min $\{\delta_1, \delta_2\}$

Then $\delta > 0$

Now $] p - \delta, p + \delta [\not\subset] p - \delta_1, p + \delta_1 [\subseteq G_1$

$] p - \delta, p + \delta [\subseteq] p - \delta_2, p + \delta_2 [\subseteq G_2$

$\therefore$ $] p - \delta, p + \delta [\subseteq G_1 \cap G_2$

$\therefore$ p is an interior point of $G_1 \cap G_2$

$\therefore$ $G_1 \cap G_2$ is an open set.

Theorem 2.12: The intersection of a finite number of open sets in R is open.

Proof : Let G_1, G_2, ..., G_n be n open sets in R.

Let $G = G_1 \cap G_2 \cap G_3 \cap .... \cap G_n$

If G = ϕ, there is nothing to prove. ϕ is an open set.

We assume that $G \neq \phi$

Let $p \in G$

$\Rightarrow$ p $\in$ G_1, for each i = 1, 2, 3, ..., n

Now p $\in$ G_1, and G_1 is open.

$\therefore$ p is an interior point of G_1

$\therefore$ There exists $\delta_1 > 0$ such that

$$]\, p-\delta_1,\ p+\delta_1\, [\leq G_1$$

Similarly, p is an interior point of G_2.

$\therefore$ There exists $\delta_2 > 0$ such that

$$]\, p-\delta_2,\ p+\delta_2\, [\subseteq G_2$$

In this way, p is an interior point of G_n.

$\therefore$ There exists $\delta_n > 0$ such that

$$]\, p-\delta_n,\ p+\delta_n\, [\subseteq G_n$$

Let $\delta = \min\ \{\delta_1, \delta_2, ..., \delta_n\}$

Then $\delta > 0$

Now $]p-\delta,\ p+\delta[\subseteq]p-\delta_1, p+\delta_1[\subseteq G_1$

$]p-\delta,\ p+\delta[\subseteq]p-\delta_2, p+\delta_2[\subseteq G_2$

$]p-\delta,\ p+\delta[\subseteq]p-\delta_n, p+\delta_n[\subseteq G_n$

$\therefore$ $]p-\delta,\ p+\delta[\leq G_1 \cap G_2 \cap .... \cap G_n = G$

$\therefore$ p is an interior point of G.

$\therefore$ G is an open set.

Example 1 : Examine if the intersection of the family of open sets $G_n =]-\frac{1}{n}, \frac{1}{n}[$, $n \in N$ is open.

Solution : We know that, for each $n \in N$, $]-\frac{1}{n}, \frac{1}{n}[$ is open

Now, $G_1 =]-1, 1\ [$

$G_2 =]-\frac{1}{2}, \frac{1}{2}[$

$G_3 =]-\frac{1}{3}, \frac{1}{3}[$

$G_n =]-\frac{1}{n}, \frac{1}{n}[$

Each G_i is open.

$\bigcap_{i=1}^{\infty} G_i = \{0\}$, which is not open.

Example 2 : Examine if the intersection of the family of open sets

$G_n =]-n, n[$, $n \in \underline{\underline{N}}$ is open.

Solution: We know that, for each $n \in \underline{\underline{N}}$,

] – n, n [is open.

Now $G_1 =]-1, 1[$

$G_2 =]-2, 2[$

$G_3 =]-3, 3[$

.....

.....

$G_n =]-n, n[$

.....

Each G_i is open

$\bigcap_{i=1}^{\infty} Gi =]-1, 1[= G_1$, which is open.

Theorem 2.13: Let S be a subset of R. Then int S is an open set.

Proof: If int S = ϕ, there is nothing to prove.

ϕ is an open set.

We assume that int $S \neq \phi$

Let $p \in$ int S

$\therefore$ p is an interior point of S.

$\therefore$ There exists a nbd N(p) of p such that $N(p) \subseteq S$.

Let q $\in$ N(p)

$\therefore$ N(p) is also a nbd of q.

$\therefore$ $N(p) \subseteq S$, q is an interior point of S.

Thus q $\in$ N(p) $\Rightarrow$ q $\in$ int S

$\therefore$ $N(p) \subseteq$ int S

$\therefore$ p is an interior point of int S

$\therefore$ int S is an open set.

Theorem 2.14 : The interior of a set S is the largest open subset of S.

Proof: We know that int S is open. Again, by definition, int $S \subseteq S$.

We are to prove that int S is the largest open subset of S.

Let T be any open subset of S.

Let $P \in T$

$\therefore$ T is open, p is an interior point of T.

$\therefore$ There exists a nbd N(p) of p such that $N(p) \subseteq T$.

But $T \subseteq S$

$\therefore$ $N(p) \subseteq S$.

$\therefore$ p is an interior point of S.

$\therefore$ $p \in$ int S

Thus $p \in T \Rightarrow p \in$ int S

$\therefore$ $T \subseteq$ int S.

Thus, the interior of S is the largest open subset of S, i.e., the interior of S contains every open subset of S.

Corollary

We have seen that the interior of S contains every open subset of S.

$\therefore$ int S = union of all open subsets of S.

Let Us Recall

The difference of two sets A and B, denoted by A – B, is the set of elements which belong to A, but which do not belong to B, i.e.,

A– B = $\{x : x \in A, x \notin B\}$

2.8 CLOSED SET

Definition 1 : Let S be a subset of R. S is said to be closed if R – S is open.

Examples :

We know that R is an open set. ϕ is the complement of R. Therefore, ϕ is a closed set.

Again, ϕ is an open set. R is the complement of ϕ. Therefore, R is a closed set.

Note:

1. R is both open and closed.
2. ϕ is both open and closed.

Theorem 2.14: Let S be a subset of R. S is closed if and only if $S' \leq S$, where S' denotes the derived set of S.

Proof: Let S be closed

$\therefore$ R – S is open.

Let $p \in R - S$

$\therefore$ p is an interior of $R - S.$

$\therefore$ There exists a nbd N(p) of p such that N(p) $\leq R - S.$

$\Rightarrow N(p) \cap S = \phi$

$\therefore$ p is not a limit point of S.

$\therefore p \notin S'$

Thus, $p \in R - S \Rightarrow p \notin S'$

This implies $p \in S' \Rightarrow p \notin R - S$, i.e., $p \in S$

Thus, $p \in S' \Rightarrow p \in S$

$\therefore S' \subseteq S$

Conversely, let S be a subset of R such that $S' \leq S$.

Let $p \in R - S$

$\Rightarrow p \notin S$

$\Rightarrow p \notin S', \quad (\because S' \subseteq S)$

$\therefore$ There exists a nbd N(p) of p such that N(p) $\cap S = \phi$

$\therefore$ N(p) $\subseteq R - S$

Thus, $p \in R - S \Rightarrow N(p) \subseteq R - S$

$\therefore$ $R - S$ is open

$\therefore$ S is closed.

In view of the above theorem, a closed set can also be defined as follows:

Definition 2 : Let S be a subset of R S is said to be closed if $S' \subseteq S$, where S' denotes the derived set of S.

Note: The definitions 1 and 2 are equivalent.

Remark: Note the following two questions.

Q 1. Define a closed set. Prove that a subset S of R is closed if and only

if $S' \leq S$.

Q 2. Define a closed set. Prove that a subset S of R is closed if and only if R – S is open.

For Q 1, you will have to write definition 1, and prove theorem 2.14

Fr Q 2, you will have to write definition 2, and prove theorem 2.14.

Example 1 : Let us consider the set N of natural numbers.

We know that N has no limit point.

$\therefore N' = \phi$. So $N' \subseteq N$

$\therefore$ N is a closed set.

2. Let us consider the set Z of integers.

We know that Z has no limit point.

$\therefore Z' = \phi$. So $Z' \subseteq Z$

$\therefore$ Z is a closed set.

3. Let us consider the set Q of rationals. We know that $Q' = R$. So $Q' \subseteq Q$

$\therefore$ Q is not closed.

Again every nbd of $p \in Q$ contains rationals and irrationals. Therefore no point of Q is an interior point of Q.

$\therefore$ Q is not open.

Note: Q is neither open nor closed.

Theorem 2.15 : The union of two closed sets in R is closed.

Proof : Let F_1, F_2 be two closed sets of R.

$\therefore F_1' \subseteq F_1, \ F_2' \subseteq F_2$

We have $(F_1 \cup F_2)' = F_1' \cup F_2'$ (thm 2.3 (b))

Now $F_1' \subseteq F_1 \subseteq F_1 \cup F_2$

$F_2' \subseteq F_2 \subseteq F_1 \cup F_2$

$\therefore F_1' \cup F_2' \subseteq F_1 \cup F_2$

$\Rightarrow (F_1 \cup F_2)' \subseteq F_1 \cup F_2$

$\Rightarrow F_1 \cup F_2$

Theorem 2.16: The union of a finite number of closed sets in R is closed.

Proof: Let F_1, F_2, ..., F_n be n closed sets in R. Let $F = F_1 \cup F_2 \cup .... \cup F_n$

Now $F_1' \subseteq F_1, F_2' \subseteq F_2, \ldots, F_n' \subseteq F_n$

We have $(F_1 \cup F_1 \cup \ldots \cup F_n)' = F_1' \cup F_2' \cup \ldots \cup F_n'$ i.e.

$F' = F_1' \cup F_2' \cup \ldots \cup F_n'$

Now $F_1' \subseteq F_1 \subseteq F$

$F_2' \subseteq F_2 \subseteq F$

.....

.....

$F_n' \subseteq F_n \subseteq F$

$\therefore F_1' \cup F_2' \cup \ldots \cup F_n' \subseteq F$

$\Rightarrow F' \subseteq F$

$\therefore F = F_1 \cup F_2 \cup \ldots \cup F_n$ is closed.

Example 3 : Examine if the union of the family of closed sets

$F_n = \left[-\frac{1}{n}, \frac{1}{n}\right], n \in N$ is closed.

Solution: Each point of $F_n, n \in N$ is a limit point of F_n. $\therefore F_n' \subseteq F_n$

$\therefore F_n$ is closed, $n \in N$.

N ow $F_1 = [-1, 1]$

$F_2 = \left[-\frac{1}{2}, \frac{1}{2}\right]$

$F_3 = \left[-\frac{1}{3}, \frac{1}{3}\right]$

.....

.....

$F_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$

Now $F_1 \cup F_2 \cup F_3 \cup \ldots \cup F_n \cup \ldots$

$= \bigcup_{i=1}^{\infty} F_i = [-1, 1] = F_1$, which is closed.

Example 4: Examine if the union of the family of closed sets

$F_n = \left[\frac{1}{n}, 4 - \frac{1}{n}\right], n \in N$ is closed.

Solution: Each point of F_n'. $\therefore F_n' \subseteq F_n$. $\therefore F_n$ is closed.

Now $F_1 = [1, 3]$

$$F_2 = \left[\frac{1}{2}, 4-\frac{1}{2}\right]$$

$$F_3 = \left[\frac{1}{3}, 4-\frac{1}{3}\right]$$

.....

.....

$$F_n = \left[\frac{1}{n}, 4-\frac{1}{n}\right]$$

Now $F_1 \cup F_2 \cup F_3 \cup .... \cup F_n \cup ....$

$= \bigcup_{i=1}^{\infty} F_i =]0, 3[$, which is not closed.

Each point of] 0, 3 [is a limit point of] 0, 3 [. Moreover, 0, 3 are limit points of] 0, 3 [. But 0, 3 do not belong to] 0, 3 [. So,]0, 3 [is not closed.

Do You Know ?

Any closed interval of the real numbers is a closed set.

Let *a, b* be reals, *a* < *b*.

[*a, b*] is a closed set.

] *a, b* [is not closed.

Theorem 2.17 : The intersection of two closed sets in R is closed.

Proof: Let F_1, F_2 be two closed sets in R. Let $F = F_1 \cap F_2$

Now, $F_1' \subseteq F_1, F_2' \subseteq F_2$

$F' = (F_1 \cap F_2)' \subseteq F_1' \cap F_2'$ (Thm 2.3 (c))

Now $F' \subseteq F_1' \cap F_2' \subseteq F_1' \subseteq F_1$

$F' \subseteq F_1' \cap F_2' \subseteq F_2' \subseteq F_2$

$\therefore\ F' \subseteq F_1 \cap F_2$, i.e., $F' \subseteq F$

$\therefore\ F = F_1 \cap F_2$

Theorem 2.18 : The intersection of a finite number of closed sets in R is closed.

Proof : Let F_1, F_2, ..., Fn be n closed sets in R. Let $F = F_1 \cap F_2 \cap ..... \cap F_n$

Now, $F_1' \subseteq F_1,\ F_2' \subseteq F_2, ..., F_n' \subseteq F_n$

$F' = (F_1 \cap F_2 \cap ..... \cap F_n)' \subseteq F_1' \cap F_2' \cap ..... \cap F_n'$

Now, $F' \subseteq F_1' \cap F_2' \cap ..... \cap F_n' \subseteq F_1$

Similarly, $F' \subseteq F_2, ..., F' \subseteq F_n$

$\therefore\ F' \subseteq F_1 \cap F_2 \cap ..... \cap F_n \subseteq F_1$, i.e., $F' \subseteq F$

$\therefore$ F is closed.

Theorem 2.19: The intersection of an arbitrary collection of closed sets in $\underline{\underline{R}}$ is closed.

Proof: Let $\{F_i : i \in I\}$ be an arbitray collection of closed sets.

Let $F = \bigcap_{i \in I} F_i$

If $F' = \phi$, there is nothing to prove.

Clearly $F' \subseteq F$, and so F is closed.

We assume that $F' \neq \phi$.

Let $p \in F'$.

$\therefore$ p is a limit point of F.

We choose $\delta > 0$

Then $]p - \delta, p + \delta[$ contains a point, say q, of F.

$q \in]p - \delta, p + \delta[\cap F$

$\Rightarrow q \in]p - \delta, p + \delta[\cap F_j$, for each $j \in I$

$\therefore$ p is a limit point of F_j, for each $j \in I$

Hence $p \in \bigcap_{i \in I} F_i$, i.e., $p \in F$

Thus, $p \in F' \Rightarrow p \in F$

$\therefore\ F' \subseteq F$

$\therefore$ F is closed.

2.9 CLOSURE OF A SET

Let S be a subset of the set of real numbers. A point $p \in \underline{\underline{R}}$ is said to be an adherent point of S if every nbd of p contains at least one point of S.

Note 1 : Clearly a point $p \in S$ is an adherent point of S. Every nbd of S contains at least one point of S, viz., the point itself.

2. If $p \notin S$, then p is an adherent point of S only if p is a limit point of S.

3. If An adherent point may or may not belong to the set.

4. An adherent point may or may not be a limit point.

Definition: The set of all adherent points of S is called the closure of S. Generally it is denoted by $\bar{S}$.

Theorem 2.20 : Let S be a subset of R. Then $\bar{S} = S \cup S'$, where S' denotes the derived set of S.

Proof : Let $p \in S$

Then every nbd of p contains p.

$\therefore$ p is an adherent point of S.

$\therefore\ p \in \bar{S}$

Thus $p \in S \Rightarrow p \in \bar{S}$

$\therefore\ S \subseteq \bar{S}$ (1)

Again, let $q \in S'$

$\therefore$ q is a limit point of S.

$\therefore$ every nbd of q contains a point of S other than q.

$\therefore\ q \in \bar{S}$

Thus, $q \in S' \Rightarrow q \in \bar{S}$

$\therefore\ S' \subseteq \bar{S}$ (2)

From (1) and (2), $S \cup S' \subseteq S'$ (3)

Again let $r \notin SUS'$

$\Rightarrow r \notin S$ and $\Rightarrow r \notin S'$

$\therefore\ r \notin S'$, there exists a nbd N(r) of r such that $N'(r) \cap S = \phi$

Again, since $r \notin S,\ N'(r) \cap S = \phi \Rightarrow N(r) \cap S = \phi$

$\therefore$ r is not an adherent point of S.

$\therefore\ r \notin \bar{S}$

Thus $r \notin S \cap S' \Rightarrow r \notin \bar{S}$

$\therefore\ \bar{S} \subseteq S \cup S'$ (4)

From (3) and (4)

$S \cup S' = \bar{S}$

Do You Know ?

The contrapositive of the statement “if p, then q” is “if not q, then not p”.

A statement and its contrapositive are logically equivalent.

Let A, B be two sets

If $x \in A \Rightarrow x \in B,$ then $A \subseteq B$ contrapositively,

if $x \notin B \Rightarrow x \notin A,$ then $A \subseteq B$

Theorem 2.21: Let S be a subset of R. Then S is closed if and only if $S = \bar{S}$.

Proof : Let S be a closed set.

$\therefore\ S' \subseteq S$

$\therefore\ S \cup S' = S$

$\Rightarrow \bar{S} = S$ (Thm 2.20)

Conversely, let S be a subset of $\underline{\underline{R}}$ such that $S = \bar{S}$.

This $\Rightarrow S = S \cup S'$

$\Rightarrow S' \leq S$

$\Rightarrow$ S is closed.

Example 5 : Find the closure of the set $S = \left\{\frac{1}{n} : n \in N\right\}$

Solution: We have $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$

We know that 0 is the only limit point of S.

$\therefore\ S' = (0)$

Thus, $\bar{S} = S \cup S'$

$$= \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\} \cup \{0\}$$

$$= \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$$

Example 6 : Find the closures of

(i) N (ii) Z (iii) Q (iv) R

Solution: (i) Let $p \in R$

No nbd of p contains of point of N other than p. $\therefore\ N' = \phi$.

$\therefore\ \bar{N} = N \cup N' = N \cup \phi = N$

(ii) We have $Z' = \phi$

$\therefore \bar{Z} = Z \cup Z' = Z \cup \phi = Z$

(iii) We have $\underline{\underline{Q'}} = \underline{\underline{R}}$

$\therefore\ \bar{Q} = Q \cup Q' = Q \cup R = R$

(iv) We have $\underline{\underline{R'}} = \underline{\underline{R}}$

$\therefore\ \bar{R} = R \cup R' = R \cup R = R$



CHECK YOUR PROGRESS 3

Q 1: Show that the set $\left\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots.\right\}$ is neither open nor closed.

Q 2: Show that the set

$S = \left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots.\right\}$ is closed but not open.

Q 3: Examine if the union ofo the family of closed sets

$F_n = \left[\frac{1}{n}, 2\right], n \in N$ is closed.

2.10 COUNTABILITY OF SETS

Definition 1 : Let S be a subset of R. S is said to be denumerable if there exists a bijective mapping $f : N \to S$. A denumerable set is also called countably infinite set.

Definition 2 : A set which is either finite or denumerable is said to be countable.

A set, which is not countable, is said to be uncountable.

Examples :

1. Let us consider the set N of natural numbers.

   Let $f : N \to N, f(n) = n$

   Let $x_1, x_2 \in N$

   Then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

   $\therefore$ *f* is one-one.

   For every *y* of N, there exists $x \in N$ such that $f(x) = y$

   $\therefore$ f is onto

   $\therefore$ f is a bijective mapping

   $\therefore$ N is countable.

2. Let us consider the set Z of integers.

   Let $f : Z \to N,$

$$f(x)=\begin{cases}\frac{1}{2}n, \text{if } n \text{ is even}\\ \frac{1}{2}(1-n), \text{if } n \text{ is odd.}\end{cases}$$

Clearly f is one-one and onto, i.e., f is a bijective mapping.

$\therefore$ Z is countable

Example 7 : Show that the set E of even natural numbers is countable.

Solution: Let $f:\mathrm{N}\to\mathrm{E},\ f(n)=2n$

Clearly f is one-one and onto.

$\therefore$ E is countable.

Note: If S is denumerable, there exists a bijective mapping f from N to S. Thus, corresponding to every element $n\in\mathrm{N}$, there exists $f(n)\in\mathrm{S}$ such that the elements of S can be described as $f(1), f(2), ..., f(n), ...$ or as $a_1, a_2, ..., a_n, ...$

Theorem 2.22: Every subset of a countable set is countable.

Proof : Let S be countable. Then S is either finite or denumerable.

Let A be a subset of S.

Case 1. Let S be finite.

Then A is also finite, and so it is countable.

Case 2. Let S be denumerable, and A be finite.

Clearly A is countable.

Case 3. Let S be denumerable, and A be infinite.

Now S can be written as

$\mathrm{S}=\{a_1, a_2, a_3, ...\}$

Let n_1 be the smallest subscript for which $a_{n_1}\in\mathrm{A}$.

We consider a new set $\mathrm{A}-\{a_{n_1}\}$.

Let n_2 be the smallest subscript such that $a_{n_2}\in\mathrm{A}-\{a_{n_1}\}$.

In this way, let n_k be the smallest subscript such that

$a_{n_k}\in\mathrm{A}-\{a_1, a_2, ..., a_{n_{k-1}}\}$

Since A is infinite, such an element a_{n_k} always exists for each $k\in\mathrm{N}$

Now A – $\{a_{n_1}, a_{n_2}, ..., a_{n_k}\}\neq\phi$, for each $k\in\mathrm{N}$

So we can write

A = $\{a_{n_1}, a_{n_2}, ..., a_{n_k}, ...\}$

We define a mapping

$f : \mathrm{N} \to \mathrm{A},\ f(k) = a_{n_k}$

Clearly f is one-one and onto.

$\therefore$ A is denumerable, and so, it is countable.

Theorem 2.23 : The union of two countable sets is countable.

Proof : Let S and T be two countable sets, where $\mathrm{S} \cap \mathrm{T} = \phi$

Case 1 : Let S and T be finite. Then $\mathrm{S} \cup \mathrm{T}$ is also finite, and so, it is countable.

Case 2 : Let S be finite and T be denumerable.

Let S = $\{a_1, a_2, ..., a_k\}$

T = $\{b_1, b_2, ..., b_n, ...\}$

We define $f : \mathrm{S} \cup \mathrm{T} \to \mathrm{N},$ by the rule

$f(a_i) = i,$ for $1 \le i \le k$

$f(b_n) = n + k,$ for each n

Clearly f is one-one and onto.

$\therefore$ $\mathrm{S} \cup \mathrm{T}$ is denumerable, and, so countable.

Case 3 : Let S and T be denumerable.

Let S = $\{a_1, a_2, a_3, ..., a_n, ...\}$

T = $\{b_1, b_2, b_3, ..., b_n, ...\}$

We define $\mathrm{S} \cup \mathrm{T} \to \mathrm{N},$ by the rule

$f(a_n) = 2n$

$f(b_n) = 2n - 1$

Clearly f is one-one and onto, and, so countable.

Note: 1. The Theorem holds good if $\mathrm{S} \cap \mathrm{T} \neq \phi$.

2. The union of a countable number of countable sets is countable.

Example 8 : Show that the set of all rational numbers between [0, 1] is countable.

Solution: Let us construct the following sets.

$\mathrm{A}_1 = \{0, 1\}$

$\mathrm{A}_2 = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\right\}$

$\mathrm{A}_3 = \left\{\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, ...\right\}$

$$A_4 = \left\{\frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, ...\right\}$$

$$A_5 = \left\{\frac{4}{5}, \frac{4}{7}, \frac{4}{9}, \frac{4}{11}, ...\right\}$$

.....

.....

We note that each of the above sets is countable.

$$\therefore \cup A_i = A_1 \cup A_2 \cup A_3 \cup A_4 \cup ....$$

$$= \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, ...\right\}$$

= [0, 1]

Which is countable.

Example 9 : Prove that the set Q of all rational numbers is countable.

Proof Case 1 : First we consider the set Q_+ of all positive rational numbers.

Let us construct the following sets.

$$A_1 = \{1, 2, 3, ...\}$$

$$A_2 = \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...\right\}$$

$$A_3 = \left\{\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, ...\right\}$$

$$A_4 = \left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, ...\right\}$$

$$A_5 = \left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{6}{5}, \frac{7}{5}, ...\right\}$$

.....

.....

We note that each of the above sets is countable. Now we list the elements as follows:

$$A_1 : 1,\ 2,\ 3,\ 4,\ ....$$

$$A_2 : \frac{1}{2},\ \frac{2}{2},\ \frac{3}{2},\ \frac{4}{2},\ ....$$

$$A_3 : \frac{1}{3},\ \frac{2}{3},\ \frac{3}{3},\ \frac{4}{3},\ ....$$

$$A_4 : \frac{1}{4},\ \frac{2}{4},\ \frac{3}{4},\ \frac{4}{4},\ ....$$

$A_5 : \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, ...$

.....

.....

Thus, we get

$$\underline{\underline{Q}}_{+} = \left\{1, \frac{1}{2}, 2, \frac{1}{3}, \frac{2}{3}, 3, \frac{1}{4}, \frac{3}{4}, \frac{4}{3}, ...\right\}$$

$$= \cup A_i, (i = 1, 2, 3, ...)$$

which is countable.

Case 2 : Next we consider the set Q_ of all negative rational numbers.

We define a mapping $f : Q_+ \to Q_-$ by the rule $f(x) = -x, \forall x \in Q_+$

Clearly f is one-one and onto.

Since Q_+ is countable, therefore Q_ is also countable.

Case 3 : Clearly {0} is countable.

$\therefore\ Q = Q_- \cup \{0\} \cup Q_+$ is countable.

Theorem 2.24 : The open interval] 0, 1 [is not countable.

Proof : Let I =] 0, 1 [

$$= \{x \in R : 0 < x < 1\}$$

If possible, let I be denumerable. Then the elements of I can be described as $a_1, a_2, a_3, ...$

We know that every rational number can be expressed as a terminating decimal or as a non-terminating but recurring decimal. On the other hand every irrational number is a non-terminating and non-recurring decimal.

Now, a terminating decimal can be expressed as a non-terminating decimal either by using 9's or by using 0's. For example 0.15 can be written as 0.14999.... or 0.15000.....

We use only 9's and ignore 0's. Then every real number in I will have a unique non-terminating decimal representation.

Let $a_1 = 0.a_{11}\, a_{12}\ a_{13}....$

$a_2 = 0.a_{21}\, a_{22}\ a_{23}....$

$a_3 = 0.a_{31}\, a_{32}\ a_{33}....$

.....

..... where $0 < a_{ij} < 9$.

We have assumed that every real number in I has a place in the enumeration $a_1, a_2, a_3, ...$

We consider the real number

$$x = 0.x_1x_2x_3....,$$

where $x_k \begin{cases} = 1, \text{ if } a_{kk} \neq 1 \\ = 2, \text{ if } a_{kk} = 1 \end{cases}$

Since *x* differs from a_i in the ith decimal place, therefore *x* is different from each a_i.

$\therefore \quad x \notin \{a_1, a_2, a_3, ...\}$

But 0 < *x* < 1, i.e., $x \in \mathrm{I}$

Now $x \in \mathrm{I}$, but $x \notin \{a_1, a_2, a_3, ...\}$

This is a contradiction.

So, our assumption is wrong.

$\therefore$ I is not denumerable, and, so, not countable.

Note: We know that if a set is countable, every subset of it is also countable. So, if a set is not countable, its superset is also not countable.

Since I =] 0, 1 [is not countable, therefore R is also not countable.



2.11 LET US SUM UP

- A subset S of R is said to be a neighbourhood (nbd) of a real number p if there exists an open interval] *a, b* [such that $\mathrm{p} \in]\mathrm{a}, \mathrm{b}[\subseteq \mathrm{S}$
- Let S be a subset of R. A real number p is said to be a limit point of S if every nbd of p contains a point of S other than p.
- The set of all points of S is called the derived set of S. Generally it is denoted by S'.
- Every infinite bounded subset of R has a limit point of R. This result is known as the Bolzano-Weierstrass Theorem.
- Let S be a subset of R. A point $\mathrm{p} \in \mathrm{S}$ is said to be an interior point of S if there exists a nbd N(p) of p such that $\mathrm{N(p)} \subseteq \mathrm{S}$.
- The set of all interior points of S is called the interior of S. Generally it is denoted by int S.

- Let S be a subset of R. S is said to be an open set if every point of S is an interior point of S.
- The interior of a set S is the largest open subset of S.
- Let S be a subset of R. S is said to be closed if R – S is open.
- Let S be a subset of R. S is closed if and only if $S' \subseteq S$, where S' denotes the derived set of S.
- Let S be a subset of R. S is said to be closed if $S' \subseteq S$.
- R and ϕ are both open and closed.
- Q is neither open nor closed.
- Let S be a subset of R. A point $p \in R$ is said to be an adherent point of S if every nbd of p contains at least one point of S.
- The set of all adherent points of S is called the closure of S.
- A set which is either empty, finite or countably infinite is called a countable set.



2.12 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: Given that S is a nbd of a real number p.

$\therefore$ There exists an open interval of the form $]p-\delta, p+\delta[$ for same $\delta > 0$ such that

$]p-\delta, p+\delta[\subseteq S$ (1)

Again $S \subseteq T$, (2) (given)

From (1) and (2) $]p-\delta, p+\delta[\subseteq T$

$\therefore$ T is also a nbd of p.

Ans to Q No 2: Let $p \in R-S$

$\therefore p \notin S$

If possible, let S be a nbd of p.

$\therefore$ There exists an interval $]p-\delta, p+\delta[$ for same $\delta > 0$ such that

$]p-\delta, p+\delta[\subseteq S$

$\Rightarrow p \in S,$ which is a contradiction.

$\therefore$ S cannot be a nbd of any point of R – S.

Ans to Q No 3:

(a) The set of real number R is a nbd of each real number p, for any open interval containing ps is a subset of R.

(b) The set of rational numbers Q is not a nbd of any of its points p, for any open interval containing p will contain infinite number of irrational numbers. Clearly the open interval cannot be a subset of Q.

Check Your Progress 2

Ans to Q No 1: Let $p \in N$.

Every nbd of p contains points not belonging to N. So, p cannot be an interior point of N. So, int $N = \phi$.

No point of N is an interior point of N. So N is not an open set.

Ans to Q No 2: int $Z = \phi$

Z is not an open set.

Ans to Q No 3: int $Q = \phi$

Q is not an open set.

Ans to Q No 4: Every point of R is an interior point of R

R is an open set.

Ans to Q No 5:

(i) Let us consider] *a, b* [

Let $p \in]a,b[$

$\therefore a < p < b$

Let c, d be two real numbers such that

$a < c < p, \quad p < d < b$

i.e., $a < c < p < d < b$

i.e., $p \in]c,d[\subseteq]a,b[$

Thus, p is an interior point of] *a, b* [i.e., every point of] *a, b* [is an interior point of] *a,b*[

$\therefore$] *a, b* [is an open set.

(ii) Let us consider] *a, b* [

a is not an interior point of [*a, b* [

$\therefore$] *a, b* [is not an open set.

(iii) Let us consider] *a, b*]

b is not an interior point of] *a, b*]

$\therefore$] *a, b*] is not an open set.

(iv) Let us consider [*a, b*]

a, b are not interior points of [*a, b*]

$\therefore$ [*a, b*] is not an open set.

Ans to Q No 6: Let S be a nonempty finite set.

Let $S = \{x_1, x_2, ..., x_n\}$

Let $p \in S$

If p is an interior point of S, then there exists a nbd N(p) of p such that N(p) $\subseteq$ S. This is not possible, since N(p) contains infinite number of elements, whereas S is finite.

Thus, no point of S is an interior point of S.

$\therefore$ S is not an open set.

Check Your Progress 3

Ans to Q No 1: We have $S = \left\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, ...\right\}$

Every nbd of 0 contains an infinite number of points of S.

$\therefore$ 0 is a limit point.

But $0 \notin S$. $\therefore$ S is not closed.

Let $p \in S$

Every nbd of p contains points not belonging to S.

$\therefore$ no point of S is an interior point of S.

$\therefore$ S is not open.

Ans to Q No 2: –1, and 1 are the only interior points of the set. No other elements of S are interior points. Therefore, S is not open.

–1, 1 are the only limit points of the set, and $-1 \in S, 1 \in S$. $\therefore$ S is closed.

Ans to Q No 3: Each point of F_n, $n \in N$ is a limit point of F_n. $\therefore F_n' \subseteq F_n$

$\therefore$ F_n is closed, $n \in N$

Now, $F_1 = [1, 2]$

$F_2 = \left[\frac{1}{2}, 2\right]$

$F_3 = \left[\frac{1}{3}, 2\right]$

$F_4 = \left[\frac{1}{4}, 2\right]$

.....

.....

$$F_n = \left[\frac{1}{n}, 2\right]$$

Now $F_1 \cup F_2 \cup F_3 \cup ..... \cup F_n \cup ....$

$= \bigcup_{i=1}^{\infty} F_i =]0, 2],$ which is not closed.



2.13 FURTHER READING

1) Principles of Mathematical Analysis by Walter Rudin.
2) Methods of real Analysis by Richand R. Goldbery.
3) Mathematical Analysis by S. C. Malik and Savita Arora.
4) A Course of Mathematical Analysis by shanti Narayan.



2.14 MODEL QUESTIONS

Q.1: Define neighbourhood of a point. Is the arbitrary intersection of the neighbourhoods of a point a neighbourhood of the point? Justify your answer.

Q.2: Show that the set of real numbers is the derived set of the set of rational numbers.

Q.3: Prove that every open set is a union of open intervals.

Q.4: Is every infinite set open? Justify your answer.

Q.5: Let G be an open subset of R, and F a closed subset of R.
Prove that
(i) G – F is open,
(ii) F – G is closed.

Q.6: State and prove Bolzano-Weierstrass Theorem. Verify the theorem for the set

$$S = \left\{\frac{n-1}{n+1} : n \in N\right\}$$

Q.7: Show that a finite set has no limit points.

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UNIT 3 : SEQUENCE

UNIT STRUCTURE

3.1 Learning Objectives

3.2 Introduction

3.3 Sequence

- 3.3.1 Subsequence
- 3.3.2 Bounded Sequence
- 3.3.3 Monotonic Sequence

3.4 Convergence of a Sequence

- 3.4.1 Algebra of Convergent Sequence

3.5 Limit Point of a Sequence

3.6 Cantor's Theorem on Nested Intervals

3.7 Cauchy's General Principle of Convergence

3.8 Limit Superior and Limit Inferior of a Sequence

3.9 Let Us Sum Up

3.10 Answers to Check Your Progress

3.11 Further Reading

3.12 Model Questions

3.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- sequence, subsequence, bounded sequence, monotonic sequence
- convergent sequence, divergent sequence, algebra of convergent sequences
- cauchy sequence, Cauchy's General Principle of Convergence.
- cantor's Theorem on nested intervals.

3.2 INTRODUCTION

The notion of a sequence of real numbers is required to obtain the sum of an infinite number of real numbers. In this unit we will introduce the notion of sequence, subsequence and discuss the convergence,

boundedness, monotonicity of sequence. We will also discuss Cantor's theorem on nested intervals.

3.3 SEQUENCE

A function f from the set N of natural numbers to the set R of real numbers is called a real sequence, i.e., $f : N \to R$ is a real sequence.

Note : 1. Since we will deal with real sequences only, we will omit the adjective real.

2. f (1), f (2), f (3), ... are the images under f. f (n) is called the nth term (or element) of the sequence.

3. The sequence is, generally, denoted by { f (1), f (2), f (3),.....} or { f (n)} or $\{u_n\}$ or $\{s_n\}$, where $u_n = s_n = f(n)$.

4. The set consisting of all distinct elements of a sequence, without repetition, and without regard to the position of a term, is called the range of the sequence.

Examples :

1. Let $f : N \to R,\ f(n) = \frac{n}{n+1},\ n \in N$

Here $u_1 = \frac{1}{2},\ u_2 = \frac{2}{3},\ u_3 = \frac{3}{4}, \ldots\ldots\ldots$

Thus, the sequence is $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots \right\}$ i.e. $\left\{ \frac{n}{n+1} \right\}$

2. Let $f : N \to R,\ f(n) = (-1)^{n+1},\ n \in N$

Here $u_1 = 1,\ u_2 = -1,\ u_3 = 1,\ u_4 = -1\ldots\ldots\ldots$

Thus, the sequence is { 1, –1, 1, –1, ...}, i.e. $\left\{ (-1)^{n+1} \right\}$

Note :The range of the sequence is { –1, 1}.

3. Let $f : N \to R,\ f(n) \begin{cases} = -1, \text{ if } n = 1, 2 \\ = f(n-1) + f(n+2), \text{ if } n > 2 \end{cases}$

Here $u_1 = 1,\ u_2 = 1$

$u_3 = u_2 + u_1 = 2,\ u_4 = u_3 + u_2 = 3,$

$u_5 = u_4 + u_3 = 5,\ \ldots\ldots\ldots$

Thus, the sequence is { 1, 1, 2, 3, 5, 8, ...}

Note : The above sequence is known as Fibonacci sequence after the name of the Italian mathematician Fibonacci aka Leonardo Fibonacci (c 1170 – c 1250). Fibonacci was instrumental in popularising Decimal place Value System (developed in India) in through his book Liber Abaci.

4. Let $f : N \to R,\ f(n) = 1,\ n \in N$

Here $u_1 = u_2 = u_3 = \ldots\ldots\ldots = 1$

The sequence is { 1, 1, 1, ...}

Note : The above sequence is called a constant sequence.

3.3.1 Subsequence

Let us consider the following two sequences,

{ 1, 2, 3, 4, 5, ...} (1)

and {1, 3, 6, 10, 15, 21, ...} (2)

We note that the sequence (2) has been obtained from the sequence (1) by omitting some terms of (1) and without disturbing the relative positions of the remaining terms. In this case, the sequence (2) is called a subsequence of the sequence (1).

A subsequence of a sequence is defined as follows.

Let $\{u_n\}$ be a sequence. Let n_1, n_2, n_3 ..... be natural numbers such that $n < n_2 < n_3 < \ldots\ldots\ldots$ Then the sequence $\{u_{n1}, u_{n2}, u_{n3}, \ldots\ldots\}$ i.e. $\{u_{nk}\}$ is called a subsequence of $\{u_n\}$.

Example Let us consider the sequence $f : N \to R,\ f(n) = \frac{1}{n},\ n \in N$

i.e. the sequence is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots\ldots\right\}$

The sequence $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\ldots\ldots\right\}$ is a subsequence of $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots\ldots\right\}$.

3.3.2 Bounded Sequence

A sequence $\{u_n\}$ is said to be bounded above if there exists a number

M such that $u_n \le M,\ \forall n \in N$. M is said to be an upper bound of the sequence.

A sequence $\{u_n\}$ is said to be bounded below if there exists a number m such that $u_n \le m,\ \forall n \in N$. m is said to be a lower bound of the sequence.

A sequence is said to be bounded if it is both bounded above and bounded below, i.e. if $m \le u_n \le M,\ \forall n \in N$

The least upper bound (supremum) of the range is called the least upper bound of the sequence, whereas the greatest lower bound (infimum) of the range is called the greatest lower bound of the sequence.

Examples

1. Let $f : N \to R,\ f(n) = \frac{1}{n},\ n \in N$

Here the sequence is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\ldots\right\}$

lub = 1, glb = 0

The sequence is bounded

2. Let $f : N \to R,\ f(n) = (-1)^{n+1},\ n \in N$

lub = 1, glb = –1

3. Let $f : N \to R,\ f(n) = 2^n,\ n \in N$

The sequence is { 2, 2^2, 2^3, 2^4, ...}

glb = 2

There exists no upper bound.

The sequence is bounded below.

4. Let $f : N \to R,\ f(n) = -3n,\ n \in N$

The sequence is { –3, –6, –9, –12, ...}

lub = –3

There exists no lower bound.

The sequence is bounded above.

3.3.3 Monontonic Sequence

Let us consider the sequence $f : N \to R,\ f(n) = \frac{1}{n},\ n \in N$

The sequence is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\right\}$

We have $\left\{1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \ldots\ldots\right\}$

such a sequence is called a monotonically decreasing sequence.

Again, consider the sequence $f : N \to R,\ f(n) = 2^n,\ n \in N$

The sequence is { 2, 2^2, 2^3, 2^4, ...}

We note that $2 < 2^2 < 2^3 < 2^4 < .....$

such a sequence is called monotonically increasing sequence.

Definition Let $\{u_n\}$ be a sequence $\{u_n\}$ is said to be a monotonically decreasing sequence if $u_1 \geq u_2 \geq u_3 \geq u_4 \geq .....$

$\{u_n\}$ is said to be a monotonically increasing sequence if $u_1 \leq u_2 \leq u_3 \leq u_4 \leq .....$

A sequence which is either monotomically increasing or monotomically decreasing is called a monotonic sequence.

Note : Constant sequences are the only sequences which are both monotonically increasing and monotonically decreasing.

Do You Know ?

Let $a_1, a_2, ...a_n$ be n positive real numbers.

The arithmetic mean (AM) is defined by $\frac{a_1 + a_2 + ... + a_n}{n}$

The geometric mean (GM) is defined by $\sqrt[n]{a_1 a_2 ..... a_n}$

AM $\geq$ GM

The equality dolds good if $a_1 = a_2 = ...= a_n$.

Example 1. Show that the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is monotonically increasing.

Solution Here $u_n = \left(1 + \frac{1}{n}\right)^n$

$$\therefore u_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

Let us consider the (n + 1) positive numbers

$1, 1+\frac{1}{n}, 1+\frac{1}{n}, ..., 1+\frac{1}{n}$ (n times)

We know that AM > GM.

$$\therefore \frac{1+\left(1+\frac{1}{n}\right)+\left(1+\frac{1}{n}\right)+\left(1+\frac{1}{n}\right)}{n+1}$$

$$> \left(1.\ (1+\frac{1}{n})\ \ (1+\frac{1}{n})\ .....(1+\frac{1}{n})\right)^{\frac{1}{n+1}}$$

or, $\frac{1+n\left(1+\frac{1}{n}\right)}{n+1} > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$

or, $\left(\frac{1}{n+1}+1\right) > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$

or, $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$

i.e. $u_{n+1} > u_n, \forall n \in N$

Thus $u_1 < u_2 < u_3 < u_4 <$ .....

is a monotomically increasing sequence.

Theorem 3.1 Every subsequence of a monotonic sequence is also monotonic.

Proof : Let $\{u_n\}$ be a monotonic increasing sequence. Then $u_1 \le u_2 \le u_3 \le u_4 \le .....$

Let $\{u_{n_k}\}$ be a sebsequence of $\{u_n\}$. Then $n_1 < n_2 < n_3 <$ .....

Hence $u_{n_1} < u_{n_2} < u_{n_3} <$ .....

$\therefore \{u_{n_k}\}$ is monotonic increasing similarly, we can prove for a monotomic decreasing sequence.

3.4 CONVERGENCE OF A SEQUENCE

First we define limit of a sequence.

Let $\{u_n\}$ be a sequence. A real number is said to be a limit of $\{u_n\}$ if for every $\varepsilon > 0$, there exists a positive integer m, which depends upon the choice of ε, such that for every $n \in \underline{\underline{N}}$, $|u_n - L| < \varepsilon$, for all $n \geq m$.

i.e. $L - \varepsilon < u_n < L + \varepsilon$, for all $n \geq m$. Now we define a convergent sequence.

A sequence $\{u_n\}$ is said to be convergent if it has a limit $L \in R$. We say that the sequence converges to L. We write $\lim u_n = L$ or $\lim_{n \to \infty} u_n = L$

Note : A sequence is said to be divergent if it is not convergent.

Example 2 : Examine the convergence of the sequence $\left\{\frac{1}{n}\right\}$

Solution : Intuitively we have $\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ The sequence converges to 0.

Now, let us apply the definition to examine the convergence.

We choose $\varepsilon > 0$.

We know, by Archimedean property of real numbers, that there exists a positive integer m such that $0 < \frac{1}{m} < \varepsilon$.

Thus, $0 < \frac{1}{n} < \varepsilon, \ \forall n \geq m$

$\therefore \ |\frac{1}{n} - 0| < \varepsilon, \ \forall n \geq m$

$\therefore \lim_{n \to \infty} \frac{1}{n} = 0$

$\therefore$ The sequence converges to 0.



CHECK YOUR PROGRESS

Q.1: Examine the convergence of the sequence $\left\{\frac{1}{n^p}\right\},\ p>0$

Q.2: Examine the convergence of the sequence $\left\{\frac{2n+1}{n+1}\right\}$.

Theorem 3.2. If a sequence is convergent, then its limit is unique.

Proof : If possible, let a sequence $\{u_n\}$ converge to two distinct limits, say L_1 and L_2.

Then given $\varepsilon > 0$, there exists a positive integer m such that

$$|u_n - L_1| < \frac{1}{2}\varepsilon,\ \forall n \geq m$$

$$|u_n - L_2| < \frac{1}{2}\varepsilon,\ \forall n \geq m$$

Now $|L_1 - L_2| = |L_1 - u_n + u_n - L_2|$

$$\leq |L_1 - u_n| + |u_n - L_2|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

$\therefore$ $|L_1 - L_2|$ is less than any positive ε, however small.

This $\Rightarrow$ $L_1 - L_2 = 0$, i.e. $L_1 = L_2$.

Thus, the limit is unique.

Theorem 3.3 A convergent sequence is bounded.

Proof : Let the sequence $\{u_n\}$ converge to L.

Then given $\varepsilon > 0$, there exists a positive integer m such that

$$|u_n - L| < \varepsilon,\ \forall n \geq m$$

$$\Rightarrow L - \varepsilon < u_n < L + \varepsilon,\ \forall n \geq m$$

Let b = min $\{l - \varepsilon,\ u_1,\ u_2,\ u_3, \ldots\ldots,\ u_{m-1}\}$

B = max $\{l + \varepsilon,\ u_1,\ u_2,\ u_3, \ldots\ldots,\ u_{m-1}\}$

$\therefore b \leq u_n \leq B,\ \forall n$

$\therefore \{u_n\}$ is bounded.

Note 1. We have see that if a sequence is convergent, then it is bounded contrapositively, if a sequence is not bounded, then it is not convergent.

2. The converse of Theorem 3.3 is not true, i.e., if a sequence is bounded, then it may not be convergent.

Let us consider the sequence $\{(-1)^n\},\ n \in N$

Clearly, the sequence is bounded.

lub = 1, glb = 1

If possible, let $\{(-1)^n\}$ converge to L. i.e. $\lim_{n \to \infty} (-1)^n = L$

We choose ε=1. Then there exists a positive integer m such that

$$|(-1)^n - L| < 1,\ \forall n \geq m$$

This $\Rightarrow |(-1)^{2m} - L| < 1,$ and $|(-1)^{2m+1} - L| < 1$

i.e. $|1 - L| < 1,$ and $|-1 - L| < 1$

i.e. $|1 - L| < 1,$ and $|-1 + L| < 1$

$$\therefore 2 = |1 - L + 1 + L|$$
$$\leq |1 - L| + |1 + L|$$
$$< 1 + 1 = 2$$

which is not possible

$\therefore$ our assumption is wrong.

$\therefore \{(-1)^n\}$ is not convergent.

Theorem 3.4 A monotonic increasing sequence, if bounded above, is convergent, and it converges to the least upper bound.

Proof : Let $\{u_n\}$ be a monotonic increasing sequence, bounded above.

Let B be the lub.

$$\therefore u_n \leq B,\ \forall n \in N$$

For a given $\varepsilon > 0$, there exists a natural member m such that $u_m > B - \varepsilon$

$\because \{u_n\}$ is a monotomic increasing sequence,

$\therefore B - \varepsilon < u_m \leq u_{m+1} \leq u_{m+2} \leq \ldots\ldots\ldots \leq B$

i.e. $B - \varepsilon < u_n \leq B + \varepsilon,\ \forall n \geq m$

$\therefore$ The sequence is convergent, and it converges to B.

Theorem 3.5 A monotonic decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound.

Proof : Proceed as above.



CHECK YOUR PROGRESS 2

Q.1: Let the sequence $\{u_n\}$ converge to L. Show that the sequence $\{|u_n|\}$ to also converges to L. Is the converse true? Justify your answer.

Q.2: Let the sequence $\{u_n\}$ converge to L, and $u_n \le B$, for every n. Show that $L \le B$.

Q.3: Let the sequence $\{u_n\}$ converge to L. Show that every subsequence of $\{u_n\}$ also converges to L.

3.4.1 Algebra of Convergent Sequences

Definition Let $\{u_n\}$, and $\{u_n\}$ be two sequences of real numbers. Let c be a real number.
Then

(i) $\{u_n\} \pm \{\upsilon_n\} = \{u_n \pm \upsilon_n\}$

(ii) $c\{u_n\} = \{cu_n\}$

(iii) $\{u_n\} = \{\upsilon_n\} = \{u_n \upsilon_n\}$

(iv) $\left\{\frac{u_n}{\upsilon_n}\right\} = \left\{\frac{u_n}{\upsilon_n}\right\}$, provided $\{u_n\}$ is a sequence of non zero real numbers.

Theorem 3.6 Let $\{u_n\}$ and $\{u_n\}$ be two convergent sequences converging to u and v respectively. Then

(i) $\lim (u_n + \upsilon_n) = u + \upsilon$

(ii) $\lim (cu_n) = cu$, where c is a real number

(iii) $\lim u_n \upsilon_n = u\upsilon$

(iv) $\lim u_n \upsilon_n = \frac{u}{\upsilon}$, provided $\{v_n\}$ is a sequence of non zero real numbers, and $\upsilon \neq 0$

Proof : (i) $\because \{u_n\}$ converges to u,

$\therefore$ given $\varepsilon > 0$, there exists a positive integer m, such that

$$|u_n - u| < \frac{1}{2}\varepsilon, \ \forall n \ge m_1$$

Again, $\because \{\upsilon_n\}$ converges to υ,

$\therefore$ given $\varepsilon > 0$, there exists a positive integer m_2 such that

$$|\upsilon_n - \upsilon| < \frac{1}{2}\varepsilon, \ \forall n \geq m_2$$

Now $|(u_n + \upsilon_n) - (u + \upsilon)|$

$= |(u_n - u) + (\upsilon_n - v)|$

$\leq |u_n - u| + |\upsilon_n - \upsilon|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \ \forall n \geq m$

where $m = \max\{m_1, m_2\}$

$\therefore \{u_n + \upsilon_n\}$ converges to u + υ

i.e. $\lim(u_n + \upsilon_n) = u + \upsilon$

ii) If c = 0, there is nothing to prove we assume that c $\neq$ 0.

We have $|cu_n - cu| = |c||u_n - u|$ (1)

$\therefore \{u_n\}$ converges to u, given $\varepsilon > 0$, there exists a positive integer m such that

$$|u_n - u| < \frac{\varepsilon}{|c|}, \ \forall n \geq m \quad (2)$$

$\therefore$ from (1) and (2),

$\therefore |cu_n - cu| = < \varepsilon, \ \forall n \geq m$

$\therefore \{cu_n\}$ converges to cu,

i.e. $\lim cu_n = cu$

(iii) We have $|u_n + \upsilon_n - u\upsilon|$

$= |u_n(\upsilon_n - \upsilon) + \upsilon(u_n - u)|$

$\leq |u_n||\upsilon_n - \upsilon| + |\upsilon||u_n - u|$ (1)

We know that a convergent sequence is bounded. $\therefore \{u_n\}$ is convergent, so it is bounded.

$\therefore$ There exists a positive number B_i such that $|u_n| < B$, for all $n \in N$ (2)

Let k = max { B, $|\upsilon|$} (3)

From (1), (2), and (3)

$|u_n\upsilon_n - u\upsilon|$

$< k \ |\upsilon_n - \upsilon| + k|u_n - u|$ (4)

Since $\{u_n\}$, and $\{\upsilon_n\}$ converge to u and υ respectively. Therefore, for every $\varepsilon > 0$, there exist positive integers m_1 and m_2 such that

$|u_n - u| < \frac{\varepsilon}{2k}$, for all $n \geq m_1$

$|\upsilon_n - \upsilon| < \frac{\varepsilon}{2k}$, for all $n \geq m_2$

Let m = max $\{m_1, m_2\}$

$|u_n - u| < \frac{\varepsilon}{2k}$, for all $n \geq m$

$\therefore |\upsilon_n - \upsilon| < \frac{\varepsilon}{2k}$, for all $n \geq m$

$\therefore$ from (4)

$|u_n\upsilon_n - u\upsilon| < k\frac{\varepsilon}{2k} + k\frac{\varepsilon}{2k} = \varepsilon$, for all $n \geq m$

$\therefore \{u_n\upsilon_n\}$ converges to $u\upsilon$

i.e., $\lim u_n\upsilon_n = u\upsilon$

(iv) First, we will prove that there exists a positive integer m such that

$|\upsilon_n| > \frac{1}{2}|\upsilon|$, for all $n \geq m_1$

We have $\upsilon = \upsilon - \upsilon_n + \upsilon_n$

$\therefore |\upsilon| = \upsilon - \upsilon_n + \upsilon_n|$

Now we can choose m_1 so that

$|\upsilon - \upsilon_n| = |\upsilon_n - \upsilon|$

$< \frac{1}{2}|\upsilon|$, for all $n \geq m_1$ $(\because \lim \upsilon_n = \upsilon)$

$\therefore$ from (1), $|\upsilon| < \frac{1}{2}|\upsilon| + |\upsilon_2|$

or, $\frac{1}{2}|\upsilon| < |\upsilon_n|$

or $|\upsilon_n| > \frac{1}{2}|\upsilon|$, for all $n \geq m_1$

or, $\frac{1}{|\upsilon_n|} < \frac{1}{\frac{1}{2}|\upsilon|}$ for all $n \geq m_1$ (1)

Next, we prove that $\lim \frac{1}{\upsilon_n} = \frac{1}{\upsilon}$

Since $\lim \upsilon_n = \upsilon$, $\therefore$ given $\varepsilon > 0$, there exists a positive integer m_2 such that

$|\upsilon_n - \upsilon| < \frac{1}{2}|\upsilon|^2 \varepsilon$, for all $n \geq m$

Let, m = max {m_1, m_2}

Now $\left|\frac{1}{\upsilon_n} = \frac{1}{\upsilon}\right| = \left|\frac{\upsilon - \upsilon_n}{\upsilon_n \upsilon}\right|$

$$= \frac{|\upsilon_n - \upsilon|}{|\upsilon_n||\upsilon|}$$

$$< \frac{\frac{1}{2}|\upsilon|^2 \varepsilon}{|\upsilon||\frac{1}{2}\upsilon|} = \varepsilon, \text{ for all } n \geq m$$

$$\therefore \lim \frac{1}{\upsilon_n} = \frac{1}{\upsilon}$$

Now, we prove that $\lim \frac{u_n}{\upsilon_n} = \frac{1}{\upsilon_n}$

We have $\lim \frac{u_n}{\upsilon_n} = \lim \left(u_n \frac{1}{\upsilon_n}\right)$

$$= (\lim u_n)\ (\lim \frac{1}{\upsilon_n}), \text{ By (iii)}$$

$$= u\frac{1}{\upsilon} = \frac{u}{\upsilon}$$

$\therefore \left\{\frac{u_n}{\upsilon_n}\right\}$ converges to $\frac{u}{\upsilon}$

i.e. , $\lim \frac{u_n}{\upsilon_n} = \frac{u}{\upsilon}$



CHECK YOUR PROGRESS 3

Q. 1: Write dow two divergent sequences $\{u_n\}$ and $\{\upsilon_n\}$ such that $\{u_n + \upsilon_n\}$ is convergent.

Q. 2: Write down one divergent sequence $\{u_n\}$ and a convergent sequence $\{u_n\}$ such that $\{u_n \upsilon_n\}$ is convergent

Do you Know ?

Sandwich is an item of food consisting of two pieces of bread with a filling between them. It is said to be named after the fourth Earl of Sandwich, John Montagu (1718-1792) "who ate a snack of this kind in order not to have to leave the gaming table." He was not a mathematician.

Another meaning : (noun) anything in a similar arrangement (adj) of the nature of, relating to or resembling, a sandwich (verb) to lay or place between two layers.

3.5 LIMIT POINT OF A SEQUENCE

A real number L is said to be a limit point of a sequence $\{U_n\}$, if every neighbourhood of L contains an infinite number of members of the sequence,

i.e., L is a limit point of $\{U_n\}$ if given $\in>0$, however small,

$U_n \in]L-\in, L+\in[$, for an infinite number values of n, i.e.,

$|U_n - L| < \in$, for an infinite number of values of n.

Clearly, L is not a limit point of $\{U_n\}$, if there exists $\in>0$ such that $U_n \in]L-\in, L+\in[$ for at most a finite number of values of n.

Note :

1. A convergent sequence has only one limit point, its limit.
2. Any bounded sequence has at least one limit point.
3. If a baunded sequence is not convergent. Then it has at least two limit points.
4. A limit point of the range set of a sequence is also a limit point of the sequence.

   The converse of the above result is not necessarily true.

Example : Let us consider the sequence $\{U_n\}$, where $U_n = 1$.

For every $\in>0$, $U_n = 1 \in]1-\in, 1+\in[$, $\forall\, n \in N$

$\therefore$ 1 is a limit point of the sequence.

Let $\alpha \in R$, and $\alpha \neq 1$. Then for all n, $|U_n - \alpha| = |1-\alpha| \not< \in$.

Thus, there is no limit point other than 1, i.e., 1 is the only limit point

of the sequence.

We note that 1 is not a limit point of the range {1}. The range {1} has no limit point.

Example : Let us consider the sequence $\{U_n\}$, where $U_n = \frac{1}{n}$.

For every $\in > 0$, $U_n = \frac{1}{n} \in]\, 0 - \in, 0 + \in [$, $\forall\, n \in \underline{N}$

$\therefore$ 0 is a limit point point of the sequence.

We note that 0 is also a limit of the range $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$

Example : Let us consider the sequence $\{U_n\}$, where $U_n = 1+(-1)^n$.

The sequence is

{ 0, 2, 0, 2, 0, 2, ... }

0 and 2 are the limit points of the sequence.

The range {0, 2} has no limit point.

Example : Let us consider the sequence $\{U_n\}$, where $U_n = (-1)^n$, $n \in \underline{N}$.

The sequence is {–1, 1, –1, 1, –1, 1, ...}

–1, and 1 are the limit points of the sequence.

The range {–1, 1} has no limit point.

Theorem : (Bolzano - Weierstrass) Every bounded sequence has a limit point.

Proof : Let $\{U_n\}$ be a bounded sequence.

Let $A = \{U_n : n \in N\}$ be the range.

Since $\{U_n\}$ is bounded, A is also bounded.

Now, there are two possibities,– either A is finite or A is infinite.

Case 1 : Let A be finite.

Then there must exist at least one member $\alpha \in A$ such that Un = α for an infinite number of values of n.

Thus, every neighbourhood $]\alpha - \in, \alpha + \in [$, $\in > 0$, of contains $Un (= \alpha)$ for an infinite number of values of n.

$\therefore$ α is a limit point of the sequence {Un}.

Case 2 : Let A be infinite.

Now, A is bounded.

$\therefore$ by Bolzano – Weierstrass theorem for sets, A has at least one limit point. Let it be β.

Now, β is a limit point of A.

$\therefore$ every neighbourhood $]\beta-\in, \beta+\in[$, $\in>0$, contains an infinite number of members of A,

i.e., $U_n \in]\beta-\in, \beta+\in[$ for an infinite number of values of n.

$\therefore$ β is a limit point of the sequence $\{U_n\}$.

3.6 CANTOR'S THEOREM ON NESTED INTERVALS

Let us consider the intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $I_3 = [a_3, b_3]$, ... such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq$....., such intervals are called nested intervals.

The family

Example Let us consider the intervals

clearly

is a family of nested intervals.

Theorem 3.7. (Sandwich Theorem)

Let , ,

positive interger such that , for all

If , converge to L,

Proof The sequences ,

given and m_2 such that

for all

fro all

Let $m_3 = \max \{m_1, m_2\}$

Then for all

for all

Let $k = \max \{m_3, m\}$

Then for all

i.e. for all

The sequence

Note the sequence is sandwiched between ,

Theorem 3.8 Cantor's Theorem on nested intervals

Let

such that

(i) , for all

(ii)

Then

Proof : We have

for all

Again , for all

is a monotonic increasing sequence, bounded above.

is convergent (Thm 3.4)

Agian, .

is convergent (Thm 3.5)

Let lim an = a, lim b_n = b

Now lim (b_n–a_n) = 0, given

b – a = 0

i.e. b = a = s, say.

s is the least upper bound of the sequences

lower bound of the sequence

and for all

for all

i.e.

Uniqueness If possible, let then for all

We define a sequence $_n$ – t for all

lim u_n = t

Now for

Also

by Sandwich Theorem, lim u_n = s

Thus, t = s

Note : 1. A nested sequence of open and bounded intervals may not have a nonempty intersection.

Let us consider the nested sequence of open intervals

Each of these intervals is bounded.

But

2. A nested sequence of closed and unbounded intervals may not have a nonempty intersection.

Remark : Georg Cantor (1895–1918) was a celebrated mathematician. He is regarded as the father of Set Theory.

3.7 CAUCHY'S GENERAL PRINCIPLE OF CONVERGENCE

Theorem 3.9 A sequence is said to be convergent if given there exists a positive integer m such that

m, n > k.

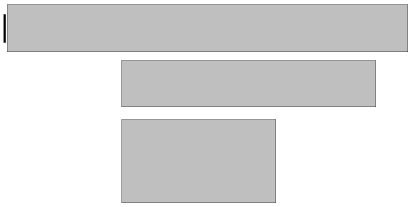
Proof Let the sequence

Let be given Then

Now for , there exists a positive integer m such that

m.

Let n, k > m. Then



i.e. 

Note : In view of the above theorem, we get the notion of Cauchy sequence after the name of the renowned Frech mathematician Augustin-Louis Cauchy (1789-1857).

Cacuchy Sequence

A sequence 
(or fundamental sequence) if, given 
such that 

Example Let us consider the sequence 

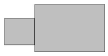
Here 

For any two positive integers n, and k (n > m, k > m, n > k), we have



Let  be given. Taking , we get 

, for all n> m, k > m

 is a Cauchy sequence.

Theorem 3.10 A Cauchu sequence is bounded.

Proof Let  be a Cauchy sequence. Then given 
positive integer m such that

, for n> m, k > m.

We choose 

, for n> m, k > m.

Taking k = m + 1, we get

, for n> m

for n> m

Let B = max

for all

Do You Know

"Every bounded sequence of real numbers contains a convergent subsequence."

This result is also know as Bolzano-Weierstrass Theorem.

Theorem 3.11 A Cauchy sequence is convergent.

Proof Let

Therefore, it must have a convergent subsequence. Let it be

Let it converge to p.

Let .

Then

There exists a positive integer q such that for any r > q, we have

Again

There exists a positive integer k such that for k>m, and any n > m, we have

Now, for an n > m, we choose r so large that n_r > m, and r > q. Then

Also is satisfied for k = n_r

Thus, for n > m

Thus, given there exists m such that for any n > m,

The Cauchy sequence

Note : Theorems 3.9, 3.10, 3.11 together constitute the Cauchy's General Principle of Convergence which may be stated as follows.

A necessary and sufficient condition forthe convergence of a sequence is that for every

such that

, for n, k > m (n > k).

Do You Know ?

A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

3.8 LIMIT SUPERIOR AND LIMIT INFERIOR OF A SEQUENCE

Let $\{u_n\}$ be a sequence, which is not necessarily bounded.

The limit of superior of the sequence $\{u_n\}$ is defined by

We denote limit superior of $\{u_n\}$ by or by

NB : Limit superior, and limit inferior of a real sequence exist uniquely (finite or infinite).

Properties (without proof)

1. For any sequence $\{u_n\}$,

2. For any sequence $\{u_n\}$,

Examples :

1. Let us consider the sequence $\{u_n\}$, where $u_n = (-1)^n$.

Now inf $\{u_n, u_{n+1}, u_{n+2}, ...\}$

= inf $\{(-1)^n, (-1)^{n+1}, (-1)^{n+2}, ...\}$

= – 1

$\sup \{u_n, u_{n+1}, u_{n+2}, ...\}$

$= \sup \{(-1)^n, (-1)^{n+1}, (-1)^{n+2}, ...\}$

$= 1$



$= -1$



$= 1$

2. Let us consider the sequence $\{u_n\}$, where $u_n = 1 + (-1)^n$

Now inf 



$= 0$



$=$ 

$= 2$

.

3. Let us consider the sequence $\{u_n\}$, where 

Now 







$= 0$

Similarly, 

4. Let us consider the sequence $\{u_n\}$,

where $u_n = n$

Now 

=n

Similarly,

5. Let us consider the sequence $\{u_n\}$,

where

Now

= –1

Similarly



3.9 LET US SUM UP

- A function f from the set N of natural numbers to the set R of real numbers is called a (real) sequence.
- A sequence is said to be bounded above if there exists a number M such that A sequence bounded below if there exists a number m such that A sequence is said to be bounded if it is both bounded above and bounded below.
- Let be a sequence. decreasing sequence if

is said to be a monotomically increasing sequence if

A sequence which is either monotonically increasing or monotonically decreasing is called a monotonic sequence.

- A sequence is said to be convergent if it has a limit

  A sequence is said to be divergent if it is not convergent.
- A convergent sequence is bounded.
- Let

  that

  i) for all

  ii)

  Then
- A necessary and sufficient condition for the convergence of a sequence is that for every

  that
- A number p is said to be a limit point of a sequence

  nbd of p, u_n belongs to the same for an infinite number of values of n.

  Thus, p is a limit point of

  ,

  for an infinite number of values of n.



3.10 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress1

Ans to Q No 1: Intuitively, we have

We are to prove that, given any

m such that for all

Now , when

i.e.

i.e. , i.e..

We take , which depends upon

for all

The sequence converges to 0.

Ans to Q No 2: Try yourself.

The give sequence is convergent. It converges to 2.

Check Your Progress 2

Ans to Q No 1: Given that the sequence

given

, for all

Now

<, for all

The sequence

The sequence

2nd part :

The converse of the above result is not convergent.

Let us consider , where $u_n = (-1)^n$. i.e. the sequence in {–1, 1, –1, 1, ...} clearly

Now the sequence

Ans to Q No 2: If possible, let L > B

We take

converges to L, there exists a positive integer m such that

, for all

or, L – $_n$ < L+ , for all

This $_n$ $_n$>B, which is a contradiction.

our assumption is wrong.

L

Ans to Q No 3: Given that the sequence

Let , i.e., be a subsequence of

Here $n_1 < n_2 < n_3 <$ .....

Let

There exists a positive integer m such that

, for all

$n_1 < n_2 < n_3 <$ ..... is an increasing sequence of natural numbers, there is a positive integer i such that are all

, for

is convergent.

Check Your Progress 3

Ans to Q No 1: Let us consider the sequence $_n$ = n, and the sequence where

Clearly the sequences , and sequence

We note that , i.e. {0, 0, 0, ...} is a constant sequence, and it converges to 0.

Ans to Q No 2: Let us consider the sequence , where u_n = n, and the sequence , where .

Clearly, the sequence is divergent. On the other hand, is convergent, and it converges to 0.

Now the sequence convergent. It is a constant sequence, and it converges to 1.

3.11 FURTHER READING

1) Principles of Mathematical Analysis by Walter Rudin

2) Methods of real Analysis by Richand R. Goldbery

3) Mathematical Analysis by S. C. Malik and Savita Arora

4) A Course of Mathematical Analysis by shanti Narayan.

3.12 MODEL QUESTIONS

Q.1: Examine if the sequence , where

Q.2: Examine if the sequence , where convergent.

Q.3: Show that a convergent sequence has only one limit point.

Q.4: Show that the sequence , where

Q.5: Prove that a convergent sequence is bounded.

*** ***** ***

UNIT 4 : INFINITE SERIES I

UNIT STRUCTURE

4.1 Learning objectives

4.2 Introduction

4.3 Infinite Series

4.4 Convergence and Divergence of Inifite Series

4.5 Two Special Series

4.6 Cauchy's Principle of Convergence for Series

4.7 Series of Positive Terms

4.8 Comparison test for Convergence of Positive Term Series

4.9 p-series

4.10 Let Us Sum Up

4.11 Answers to Check Your Progress

4.12 Further Reading

4.13 Model Questions

4.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- infinite series, convergence and divergence of infinite series
- Cauchy's Principle of Convergence, for series
- Comparison test for convergence
- divergence of series
- p-series.

4.2 INTRODUCTION

The study of infinite series is an indispensable part of analysis. In this unit we will define infinite series, convergence and divergence of infinite series, Cauchy's Principle of convergence for series. We will also discuss comparion test for convergence and divergence of series, p-series.

4.3 INFINITE SERIES

Let us consider the sequence

 i.e. 

From the terms of the sequence, we get a summation of the infinite number of terms of the sequence, viz,



Such an expression is called an infinite series.

Thus, we define an infinite series as follows.

If $\{u_n\}$ is a sequence of real numbers, then the expression

 (1)

is called an infinite series of real numbers, or simply, infinite series or series.

It is, generally, written as  or 

Example :  is an infinite series.



From the series (1), we from the sums

S_1 = first partial sum = u_1

S_2 = second partial sum = $u_1 + u_2$

S_3 = third partial sum = $u_1 + u_2 + u_3$

.....

S_n = nth partial sum = $u_1 + u_2 + u_3 + ...+ u_n$

The sequence $\{S_n\}$ is called the sequence of partial sums of the series 

4.4 CONVERGENCE AND DIVERGENCE OF INFINITE SERIES

Let us consider the series



If the sequence of partial sums $\{S_n\}$ is convergent, i.e., if there exists a number S such that 

and S is called the sum of the series.

If 

case the sum of the infinite series does not exist. Thus, we see that all infinite series cannot be summed.

Examples 1 : Let us consider the infinite series 1 + 2 + 3 + ... 

Here 

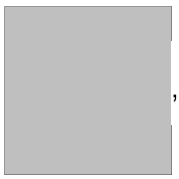
clearly , as 

The given series is not convergent. The series cannot of summed.

2. Let us consider the series



Here 

, (geometric series with common ratio 

= 



the series is convergent, and it converges to 2, i.e., 2 is the sum of the series.

3. Let us consider the series



Here 

.....

.....

i.e.

Note :

Do You Know ?

1. Addition of a finite number of terms to an infinite series does not affect the convergence or divergence.
2. Removal of a finite number of terms from an infinite series does not affect the convergence or divergence.
3. Multiplication of each term of an infinite series by a nonzero constant does not affect the convergence or divergence.

Theorem 4.1 If is convergent, then

Proof : We have



 is convergent, there exists a real number s such that



Also 

Now 

= 

= S – S =0

Note : 1. We have seen that if  is convergent, 

Contrapositively, if 

Let us consider the series 

Here 



 The series is not convergent.

2. The converse of Theorem 4.1 is not true. If , 

series may or may not converge.

Theorem 4.2 Let the series  converge to s, and the series  converge to t. Then

(i) The series 

(ii) The series  converges to , where real number.

Proof : (i) We have nth partial sum of 

=

=

=

where

Now

= s + t

The series

(ii) We have

nth partial sum of

=

=

=

Now

The series converges to

4.5 TWO SPECIAL SERIES

I. Geometric Series

Let us consider the geometric series

, where r is the common ratio.

Case (i) First we consider the case where i.e.,

Now

=

The series is convergent, and it converges to

Case (ii) Next we consider the case when

Then, there are four possibilities, viz, r = 1, r > 1, r = –1, r < –1

Let r = 1

We have

r = 1, S_n = 1 + 1 + 1 + ... + 1 (n terms)

= n

Clearly , as

The series is divergent when r = 1.

Next, let r > 1.

We have

=

Now, as (

, as

The series is divergent when r > 1

Next, let r = –1

Now

The sequence $\{S_n\}$ is divergent.

The series is divergent when r = –1

Lastly, let r < –1

Clearly the sequence $\{S_n\}$ is divergent.

The series is divergent when r < –1.

Thus, we see that the geometric series

convergent when | r | < 1, and divergent

Note : The geometric series

is convergent if | r | < 1. In this case, the sum of the series is

II. Harmonic Series

The series

is called Harmonic Series

Let

Similarly

In this way

as

The series is divergent

Note 1. H_1, H_2, H_3, ... are named harmonic numbers by Americian computer scientist and mathematician Donald Ervin Knuth (born 1938).

2. We know that if a series

. The converse of the result is not true, i.e., if ,

the series

For the harmonic series ,

But the series is not convergent.

Do You Know

"The name of the harmonic series derives from the concept of overtones or harmonics in music : The wavelengths of the overtones of a vibrating string are etc., of the string's fundamental

wavelength. Every term of the harmonic series after the first is the harmonic mean of the neighboring terms, so the terms form a harmonic progression; the phrases harmonic mean and harmonic progression likewise derive from music.''

—— Wikipedia

4.6 CAUCHY'S PRINCIPLE OF CONVERGENCE FOR SERIES

Theorem 4.3 The series converges if an only if, given , there exists a positive integer m such that

(n>k)

Proof : Let $\{S_n\}$ be the sequence of partial sums of the series

By Cauchy's Principle of convergence for sequence, $\{S_n\}$ is convergent if and only if given that

, for n, k > m (n > k)

Now $S_n - S_k$

=

=

=

The convergence of the sequence $\{S_n\}$ implies the convergence of

is convergent if and only if given s a positive integer m such that

, for n, k > m, (n>k)

4.7 SERIES OF POSITIVE TERMS

An infinite series > 0 for all n, is called a series of positive terms or a positive term series.

Theorem 4.4 A services

and only if the sequence $\{S_n\}$ of partial sums is bounded above.

Proof : Let $S_n = u_1 + u_2 + u_3 + ...+ u_n$

Then $S_{n+1} - Sn = u_{n+1} > 0$, for all n.

The sequence $\{S_n\}$ is a monotonic increasing sequence.

$\{S_n\}$ is convergent if and only if it is bounded above.

The series $\}$ is bounded above.

Note : diverges to $\}$ is not bounded above.

Let us consider the harnomic series

We have seen that

This shows that $\{H_n\}$ is not bounded above.

diverges to

Grouping of terms of a series

Let us consider the series

$1 - 1 + 1 - 1 + 1 - 1 + ...$

clearly

$S_n = u_1 + u_2 + u_3 + ...+ u_n$

$\{S_n\}$ is not convergent

The series $1 - 1 + 1 - 1 + ...$ series $(1 - 1) + (1 - 1) + ...$

i.e. $0 + 0 + ...$

Thus, we see that the grouping of terms has affected the nature of the series.

Now, let us consider the positive term series $u_1 + u_2 + u_3 + \ldots\ldots$

i.e.

Let the series

Let $S_n = u_1 + u_2 + u_3 + \ldots\ldots u_n$

clearly the sequence $\{S_n\}$ of partial sums is an increasing sequence of positive terms, bounded above.

Now ther terms of $_n\}$ be the sequence associated with the new series.

clearly $\{T_n\}$ is a subsequence of $\{S_n\}$.

$\{T_n\}$ will also converge to S.

Thus, we see that a convergent series of positive terms remains convergent under grouping.

Similarly, a divergent series of positive terms will also remain divergent under grouping.

In short, grouping does not after the behaviour of a positive term series.

4.8 COMPARISON TEST FOR CONVERGENCE OF POSITIVE TERM SERIES :

There are some tests by which the convergence of positive term series can be examined without finding the sequence $\{S_n\}$ of partial sums. It may mentioned that it is not always easy to find the sequence $\{S_n\}$ of partial sums and check its convergence. In this unit, we will discuss one test, viz, the comparison test. Other tests will be discussed in the next unit.

Theorem 4.5 Let and

I. Let there be a positive real number k (independent of n), and a positive integer m such that

(i) if is convergent,

(ii) if is divergent,

II. Suppose

and .

III. Suppose there exists a positive integer m such that

Then, (i) if is convergent,

(ii) if is divergent, them

Proof : I Let $S_n = u_1 + u_2 + u_3 + ...+ u_n$

$T_n = _1 + _2 + _3 + ... _n$

Then for all

$S_n - S_m = u_{m+1} + u_{m+2} + ... + u_n$

$= k (T_n - T_m)$

or

Putting we get

(1)

(i) Now, if $_n\}$ of partial sums is bounded above. Therefore, there exists a number

(2)

From (1) and (2), we have

The sequence $\{S_n\}$ is bounded above

is convergent.

(ii) Now, if $_n\}$ of partial sums is not bounded above. Therefore, if A is any number, however large, there exists a positive integer p such that

From (1) and (3),

or

i.e.

The sequence $\{T_n\}$ is not bounded.

is divergent.

II. We have

clearly $L > 0$.

Let be a positive number such that

, there exists a positive integer m such that

or,

or,

From (4), we have

if

Again, from (4), we have

if

Similarly, it can be proved that converges of diverges with

III. For

=

or

or

or



Now  is fixed as m is a fixed positive integer. We take 



 if  is convergent, by I, 

Also, if  is divergent, by I, 

Remark : II is frequently used in working out problems.

4.9 p-SERIES (HYERHARMONIC SERIES)

Theorem 4.6 The series



converges for p > 1, and diverges for p 

Proof : Case I Let p > 1.

Let 

Here 

Now the series  is obtained from the series  by grouping the terms as follows.



Now 









.....

In this way, 

, for all 

Now  is geometric series with common ratio 



 is convergent.

 by Theorem 4.5 {I (i)}, 

 is also convergent.

Case II Let p =1

The series is 

We have already proved that the series is divergent.

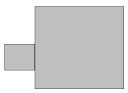
Case III Let 0 < p < 1



i.e. 

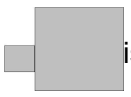
i.e.  i.e. , for all 

Now 

 is divergent (Theorem 4.5 {I (ii)}

Let 

Then 

 is not convergent.

Example 1 Examine the convergence of the series



Solution : Here

[1, 3, 5, ... is an arithemetic sequence.
nth term = 1 + (n–1) 2 = 2n – 1.

Let us form the series , where

[We note that the highest power in the numerator of u_n is n, and the highest

power in the denominator is n^3. So we take

Now, we compare the two series and

=

=

non zero real number.

=

is convergent (Theorem 4.6)

So, by comparison test,



CHECK YOUR PROGRESS 1

Q.1: Test the convergence of the following series

(i)

(ii)

(iii)

(iv) , where

(v) , where

Q.2: Show that the series is convergent, and find its sum.

Q.3: If the series of positive terms converges, show that also converges.

Q.4: Examine the convergence of the series , where



4.10 LET US SUM UP

- If $\{u_n\}$ is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + \ldots\ldots\ldots$ is called an infinite series of real numbers, or simply, infinite series or series. It is, generally, written as or
- Let us consider the series $u_1 + u_2 + u_3 + \ldots\ldots\ldots$
  If the sequence of partial sums $\{S_n\}$ is convergent, i.e., if there exists a number s such that convergent, and s is called the sum of the series.
- If is convergent, then
  If
- The geometric series $1 + r + r^2 + \ldots\ldots$
  $-1 < r < 1$.

In this case, the sum of the series is

- The harmonic series
- The p-series converges for p>1, and diverges for p

4.11 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1:

(i)

For the arithmetic sequence, 1, 5, 9, ..., nth term = 1 + (n – 1) 4 = 4n –3

For the arithmetic sequence 5, 9, 13, ..., nth term = 5 + (n – 1) 4 = 4n +1

Let us form the series , where

Now

=

=

= .

Now

 is also convergent.

(ii) Try yourself. Convergent.

(iii) Here 



Let us form the series , where 

Now 

= 

= 

Now, 

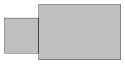
 is also divergent.

(iv) Here 

Let us form the series , where 



 is convergent. 

 is convergent.

(v) Here

Now,

=

Let us form the series , where

Now,

=

=

The series

The series

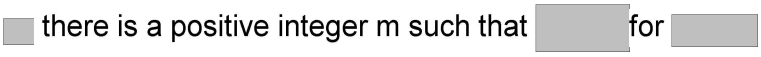
Ans to Q No 2: The series is

It is a geometric series with common ratio

The series is convergent, and its sum =

Ans to Q No 3: It is given that the series of positive terms converges.









(Theorem 4.5 I (i))

Ans to Q No 4: Here the series is 

Let us form the series , where 

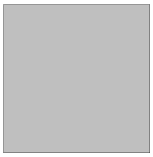
Now 

=  (Putting 

= 1, a non zero real number.

Now 

is also divergent



4.12 FURTHER READING

1) Fraleigh, John B : A first course in abstract Algebra, Pearson Education (Singapure) Pvt. Ltd. India Branch, Delhi - 11092.
2) Singh, S and Qazi, Z : Modern Algebra, Vikas Publicasting House Pvt. Ltd, New Delhi - 110014.
3) Macauley, M. : Section 6 : Fields and Galois Theory, http :// www.math.clemson.edu/~macaule/.
4) Talati, A. : Class Notes of Professor June Huh, 2022.
5) Singh, J. : Theory of Field Extension, Maharshi Dayanand University Press, Rohtal-124001.

4.13 MODEL QUESTIONS

Q 1: If the series is convergent, prove that
What happens if . Show that the series
divergent.

Q 2: Prove that the multiplication of each term of an infinite series by a non zero constant does not affect the convergence or divergence.

Q 3: Examine the convergence of the following series

(i)

(ii)

(iii) , where

(iv) , where

(v) , where

(vi) , where

[Hints :

is convergent,

*** ***** ***

UNIT 5 : INFINITE SERIES II

UNIT STRUCTURE

5.1 Learning Objectives

5.2 Introduction

5.3 Different tests of convergence

- 5.3.1 D'Alembert's Ratio Test
- 5.3.2 Cauchy's Root Test
- 5.3.3 Raabe's Test
- 5.3.4 Logarithmic Test
- 5.3.5 Cauchy's Condensation Test
- 5.3.6 De Morgam and Bertrand's Test
- 5.3.7 Gauss's Test
- 5.3.8 Cauchy's Integral Test

5.4 Aternating Series

- 5.4.1 Leibniz's Test

5.5 Absolute Convergence and Conditional Convergence

5.6 Let Us Sum Up

5.7 Answers to Check Your Progress

5.8 Further Reading

5.9 Model Questions

5.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- different tests of convergence
- alternating series, Leibniz's test
- absolute convergence and conditional convergence.

5.2 INTRODUCTION

In this unit, we will discuss some special tests of convergence. These tests will help us to determine the convergence of a wide variety of series. Furthermore, we will discuss alternating series, and Leibniz's test

for alternating series. Also, we will discuss absolute convergence and conditional convergence.

5.3 DIFFERENT TESTS OF CONVERGENCE

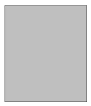
Here we will discuss eight speical tests of convergence. It will help us to determine the convergence of a wide variety of series. We will state and prove D'Alembert's Ratio Test, Cauchy's Root Test, Raabe's Test, Logarithmic Test, Cauchy's Condensation Test, De Morgan and Bertrand's Test. We will state Gauss's Test and Cauchy's Integral Test without proof.

5.3.1 D' Alembert's Ratio Test

(after the name of the French mathematician Jean le Roud d'Alembert (1717-1783) Theorem 5.1

Let  be a series of positive terms such that 

I. If L < 1, then 

II. If L > 1, them 

III. If L = 1, then test fails.

Proof I. Let L < 1.

We choose a positive number  such that L + 

, there exists a positive integer m such that

, for all 

Let L + 

Then 0 < r < 1

We have ,

,

.....





{(n –m) times}

or,  , n > m.

or, 

or, 

Let, 



Now 

q.





II. Let L > 1

We choose a positive number  such that L – 

, there exist a positive integer t such that

, for all 

Let L – 

We have 






,


or , for all 

or ,


Let 

,


Now 

1.





III. If L = 1, no conclusion can be drawn. In some cases, the series may converge. In some cases, it may diverge.

Let us consider the series 

Here 



We know that the series 

Again, let us consider the series 

Here 



We know that the series is convergent (p-series with p = 2)

Example 1. Examine the convergence of the series

, $x > 0$

Solution : Here 



= 

= x

 by d' Alembert's ratio that, the series is convergent if $x > 1$, and divergent if $x < 1$.

If $x = 1$, the series becomes 

which is convergent.

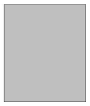
5.3.2 Cauchy's Root Test

(after the name of the French mathematician Augustin-Louis Cauchy (1789-1857).

Theorem 5.2. Let 



I. If $L < 1$, then 

II. If $L > 1$, then 

III. If $L = 1$, then the test fails.

Proof : I Let $L < 1$.

We choose a positive number  such that $L +$ 

there exists a positive integer m such that

< L + , for all

or, , for all

Let L +

Then 0 < r < 1

And , for all

Now

II. Let L > 1.

We choose a positive number such that L –

, there exists a positive integer t such that

, for all

or, , for all

Let L –

Also $P^n < u_n$, i.e. $u_n > p^n$, for all

Now

is divergent.

is also divergent (Thm 4.5I)

III. If L = 1, no conclusion can be drawn

In some cases, the series may converge. In some cases, it may diverge.

Let us consider the series and

In case f the cases

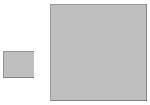
Now is divergent, whereas the series

Example 2 Examine the convergence of the series 

Solution : Here 

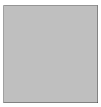






5.3.3 Raabe's Test

(after the name of the Swiss mathematician Joseph Luding Raabe (1801-1859)

Theorem 5.3 Let 



Then the series converges or diverges according as L > 1 or L < 1.

Proof : Let 

Clearly 





i.e., if 

i.e., if 

i.e., if 

i.e., if 

 The series 

is divergent if L < 1.

In this case, we take 

Note : Raabe's test is applied, when ratio test fails.

If L =1, then Raabe's test fails.

Example 3: Examine the convergence of the series



Solution: Here , for all 







 d' Alembert's ratio test fails.

Now let us apply Raabe's test

Now 

= 

= 

=

=

=

5.3.4 Logarithmic Test

Theorem 5.4 Let

Then the series converges or diverges according as L > 1 or L < 1.

Proof : Let L> 1

We take a number p such that L > P > 1

Let us consider the series , where

Now,

will be convergent if, after some particular terms,

i.e., if 

i.e., if 

i.e., if 

i.e., if 

i.e., if 

For large values of n, the left hand side tends to L, whereas the right hand side tends to p. But L > p. Hence the inequality is satisfied for sufficiently large values of n.

The given series is convergent, if L > 1.

Similarly it can be proved that the series is divergent if L < 1.

Note: 1. When the d' Alembert's ratio test fails, the above test is applied. Moreover if n occurs as an exponent in , this test should be applied.

Furthermore, if we take logarithem of 

may become easier in some cases.

2. No conclusion can be made when L = 1.

If , then the series 

converges or diverges according as p > 1 or p < 1.

Theorem 5.5

5.3.5 Cauchy's Condensation Test

Let {f(n)} be a monotonic decreasing sequence of positive real numbers and a be a positive integer greater than 1.

Then the series

f (1) + f (2) + f (3) + ...+ f (n) + ...

and a f (a) + a^2 f (a^2) + a^3 f (a^3) + ...+ a^n f (a^n) + ... converge or diverge together.

Proof : The series can be written as





The terms of the (m + 1) the group are



Number of terms in this group

= 

{ f (n)} is a monotonic decreasing sequence, f (a^{m+1}) is the smallest term in this group. Therefore,





Putting m = 0, 1, 2, ... successively in this inequality, we get





.....

.....

Adding, we get



i.e.  (1)

From (1), we see that if the series is divergent, then is also divergent.

Again, since {f (n)} is a monotonic decreasing sequence, each term in the (m+1) the group is less than f (a^m). Therefore

f (a^m +1) + f (a^m +2) + ...+ f (a^{m+1}) < a^m (a – 1) f (a^m)

Putting m = 0, 1, 2, ... successively,

We get

$f(2) + f(3) + \ldots\ldots + f(a) < (a-1) f(1)$

$f(a+1) + f(a+2) + \ldots\ldots f(a^2) < a\ (a-1) f(a)$

.....

.....

.....

Adding, we get

 (2)

From (2), we see that if the series 

 is also convergent.

Cololllary The series Let 

convergent when $p > 1$, and divergent when 

Let 

Now {log n} is an increasing sequence, and $p > 0$. Therefore





 is a monotonic decreasing sequence.

 by Cauchy's condensation test, the series 

or diverges for the same value of p as the series



i.e. 

i.e. 

Now  is a constant, and the series 

diverges according as p > 1 or p

The given series is convergent if p > 1, and divergent if

5.3.6 De Morgan and Bertrand's Test

(after the names of Augustus De Morgan (1806-1871) and Joseph Bertrand (1822-1900)

Theorem 5.6 Let

Then the series converges or diverges according as L > 1 or L < 1.

Proof : Let L > 1

We choose a number p such that L > p > 1.

Let us consider the series

Now the series

Therefore

i.e.,

i.e.

i.e.

i.e.

i.e.  terms involving nor log n in the denominator.

Now, as n becomes indefinitely large, the left hand side and the right hand side of the above inquality tend to L and p respectively. But L > p. Therefore, for large values of n, the inequality is satisfied. Therefore  is convergent.

Similarly, it can be proved that the series is divergent of L <1.

5.3.7 Gauss's Test

(after the name of the German mathematician Carl Friedrich Gauss (1777-1855). We state this test without proof. It is a very strong test. It includes both D'Alembert's Ratio Test and Raabe's Test.

Theorem 5.7 Let 

Let 

where a, b, p are real numbers, a > 0, p > 1, and $\{r_n\}$ is a bounded sequence.

Then

(i)  converges, if a > 1

(ii)  diverges, if a < 1

(iii)  converges, if a = 1, b >1

(iv)  diverges, if a = 1, b 1

5.3.8 Cauchy's Integral Test

[We state the theorem without proof].

Theorem 5.8 let f be a real - valued function with domain [1,

Let

(i) 

(ii) 

i.e. f is a nonnegative monotonically decreasing function.

(iii) The function f induce the following infinite series

f (1) + f (2) + f (3) + ... + f (n) + ...

Then the series 

converge or diverge together.

5.4 ALTERNATING SERIES

An infinite series 

consecutive terms of the series are of opposite signs.

For example 

5.4.1 Leibniz's test

(after the name of the German mathematician Gottfried Wilhelm Leibuiz (1646-1716)

Theorem 5.9 Let 

positive real numbers, and let 

series

 is convergent.

Proof : Let 

Then 





 The sequence 

Again, 







 The sequence 

Now

=

The sequence

Also,

The sequence .

The sequences and

Now

=

both the sequence and

limit.

The sequence

The series

Example 4. Let us consider the series

Here each

The sequence

The series

Leibniz's test)

5.5 ABSOLUTE CONVERGENCE AND CONDITIONAL CONVERGENCE

Let

I. If the series converges, then the series absolutely convergent.

II. If converges, but diverges, then the series is said to be conditionally convergent.

Example 5 : Let us consider the series

Here each $u_i > 0$

The sequence decreasing.

Now

The series is convergent. (by Leibniz's test)

Now let us consider the series

clearly it is a geometric series with common ratio

The series is convergent.

Hence the give series is absolutely convergent.

Remark : We will prove that if an infinite series is absolutely convergent, then it is convergent.

Example 6 : Let us consider the series

We have seen that the above series is convergent (Example 4)

Now let us consider the series 

We have seen that the above series is divergent (4.5 II Harmonic Series)

 The series 

convergent.

Theorem 5.10 An absolutely convergent series is convergent.

Proof : Let the series 

 is convergent.

Let $\{S_n\}$ be the sequence of partial sums of 



Let $\{T_n\}$ be the sequence of partial sums of 

Now, 

 The sequence $\{T_n\}$ is convergent.

 $\{T_n\}$ is Cauchy sequence.

 for any 

 for $n > m, \quad k > m$.

Let $n > k$. Now













Remark : The converse of the above theorem is not true, i.e., if a series is convergent, it may not be absolutely convergent.

Let us consider the series



We know that the series is convergent

But the series 

 The series 

convergent.

Example 7. Let us consider the series



Now let us consider the series , where 



Let us consider the series , where 

Now 

= 

The series 

 The series 

 The given series is not absolutely convergent.

Again the sequence 

decreasing.

by Leibniz's test, the series

Note : We know that if an infinite series is absolutely convergent, then it is convergent. Contrapositively, if a series is not convergent, then it is not absolutely convergent.

Do You Know ?

An infinite series of positive terms cannot be conditionally convergent.

By definition of conditional convergence, converges, but diverges. For a series of positive terms,

For a seris of positive terms,

convergence of

convergence of



CHECK YOUR PROGRESS 1

Q 1: Examine the convergence of the series

i)

ii)

iii)

iv)

v)

Q 2. Find the values of p for which the series

converges conditionally.

Q 3: Show that the series [redacted] is conditionally convergent.

Q 4: Show that the series [redacted] is convergent, and its sum is log2.

Q 5: Show that the series [redacted] is convergent, and its sum is [redacted]

Do you Know?

If the terms of absolutely convergent series are rearranged, then the series remains convergent and its sum remains unaltered.



5.6 LET US SUM UP

- Eight special tests for determining the convergence of a variey of series have been discussed.
- An infinite series [redacted] consecutive terms of the series are of opposite signs.
- Let $\{u_n\}$ be a monotonic decreasing sequence of positive real numbers, and let [redacted]. Then the alternating series [redacted] is convergent.
- If the series [redacted] converges, then the series [redacted] is said to be absolutely convergent.
- If [redacted] coverges, but [redacted] diverges, then the series [redacted] said to be conditionally convergent.
- An absolutely convergent series is convergent.

5.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: We have 











= x

 The given series is convergent if x < 1 and is divergent if x > 1 (by d'Alembert's when x = 1, the series becomes



In this case, 

Let us consider the series  where 

Now 



Now, the series

The series

The given series is convergent if

ii)

We have

(neglecting the first term)

= x

$\therefore$ the given series is covergent if x < 1, and is divergent if x > 1 (by d' Alembert's ratio test)

When x = 1, the series becomes

Here

Now, comparison test is not possible to apply.

In this case,

= 1

d' Alembert's ratio test fails.

Let us now try Raabe's test

Now 















 The series is convergent (by Raabe's test)

Thus, the given series is convergent if 

1.

(iii) The series 

We will use Cauchy's condensation test.

Here 

Now { log n} is an increasing sequence.

 log (n + 1) > log n

 (n + 1) log (n + 1) > n log n



is a monotomic decrasing sequence.

We take a = 2





 The series 

series 

 is convergent or divergent.

Now  is a constant and the series 

the series 

Thus, the given series is divergent.

(iv) 

Here 







d' Alembert's ratio test fails.

Now, we will apply Logarithmic test.

We have ,

















The given series converges if 

< 2

when p =2 , then L = 1, and the test fails.

when p =2

Now 

= 

= 

= 



The series is divergent if p = 2.

Thus, the series is convergent if p > 2, and divergent if 

(v) Here 









The given series is convergent when x < 1, and divergent when x > 1. When x = 1, 

Let us consider the series , where 

Now, 

Again the series 

The series is convergent for x = 1

Thus, the given series is convergent for , and divergent for x > 1.

Ans to Q No 2: The series is



Let p > 0

Then 

Again, 

The series is convergent for p > 0 (by Leibniz's theorem)

Let p = 0. In this case, the series becomes

1 – 1 + 1 – 1 +.....

Now 

The series is divergent for p = 0.

Let p < 0. In this case

 does not exist.

The series is divergent for p < 0.

Thus, we see that the given series is convergent only when p > 0

Again, we know that the series

 is convergent only when p > 1

The given series is conditionally convergent for 

Ans to Q No 3: The series is



clearly 

decreasing sequence.



The series is convergent.

Now let us consider the series



Here 

We take the series , where 

Now 



We know that the series 

 by comparison test, the series 

 The given series is conditionally convergent.

Ans to Q No 4: We have already proved that the series  is convergent.

Now 

 (putting x = 1)

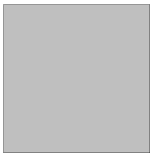
Ans to Q No 5:  is a monotonic decreasing sequence.



 The series is convergent. (by Leibniz's test)

Now 





5.8 FURTHER READING

1) Fraleigh, John B : A first course in abstract Algebra, Pearson Education (Singapure) Pvt. Ltd. India Branch, Delhi - 11092.

2) Singh, S and Qazi, Z : Modern Algebra, Vikas Publicasting House Pvt. Ltd, New Delhi - 110014.

3) Macauley, M. : Section 6 : Fields and Galois Theory, http :// www.math.clemson.edu/~macaule/.

4) Talati, A. : Class Notes of Professor June Huh, 2022.

5) Singh, J. : Theory of Field Extension, Maharshi Dayanand University Press, Rohtal-124001.

5.9 MODEL QUESTIONS

Q 1: Examine the convergence of the following series

(i)

(ii)

(iii)

(iv)

(v)

(vi)

Q 2: Discuss the convergence of the series

(i) , where

(ii) , where

Q.3: Examnine the convergence of the series

[Here

Applying d'Alembert's ratio test, we wil find that the series is convergent for $x < 1$, and divergent for $x > 1$.

For $x = 1$, Raabe's test fails. Applying De Morgan and Bertrand's test, we will find that the series is divergent for $x = 1$]

Q 4: Test the convergence of the following series

(i) 

(ii) 

Q 5: Examine if the series 

convergent.

*** ***** ***

UNIT 6 : POWER SERIES

UNIT STRUCTURE

6.1 Learning Objectives

6.2 Introduction

6.3 Power Series

6.4 Radius of Convergence, Interval of Convergence

6.5 Uniquences of power series

6.6 Properties of power series

6.7 Let Us Sum Up

6.8 Answers to Check Your Progress

6.9 Further Reading

6.10 Model Questions

6.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- notion of power series
- radius of convergence interval of convergence of power series
- uniqueness of power series
- properties of power series.

6.2 INTRODUCTION

Power series is our important type of series in analysis. They can be used to define transcendental functions such as the exponential and trigonometric functions, and many other functions. In this unit we will study power series including radius of convergence of power series, uniqueness of power series, properties of power series, etc.

6.3 POWER SERIES

An infinite series of the form



where

power series about the point c. c is called the centre of the power series.

Taking c = 0, we get the power series about 0, viz

It can be denoted by

Examples

1.

i.e. is a power series

We note that

for all real x.

2. The geometric series

i.e.

We know that the series converges for

3. The series

i.e.

We note that

, except when x = 0

i.e. the series converges only for x = 0

We note that

(i) some power series converge for all reals x, which are called everywhere convergent power series;

(ii) some power series converge for some real x, and diverge for others;

(iii) some power series converge only for x = 0, which are called nowhere convergent power series.

Convergence of power series : The convergence of a power series can be examined by using d'Alembert's ratio test or Cauchy's root test. Ratio test or Root test can be combined with the absolute convergence theorem. They be applied to general series. These two tests are named as strengthened Ratio Test and strengthened Root Test.

Strengthened Ratio Test : If

L, then the series

Strengthened Root Test : If

L, then the series

No conclusion can be drawn if L = 1. If L =



CHECK YOUR PROGRESS 1

Q 1: Examine the convergence of the following power series

(i)

(ii)

6.4 RADIUS OF CONVERGENCE : INTERVAL OF CONVERGENCE

Theorem 6.1 If the power series

than it converges absolutely for all x with

Proof : Since the power series converges for

therefore its terms converge to zero. Therefore, they are bounded and there exists

, for n = 0, 1, 2, ...

If

Since r < 1,

Therefore, by comparison test, the power series absolutely convergent.

Thus, if the power series converges for ,

then it converges absolutely for all x with

Theorem 6.2 For any power series

three posibities holds good.

I. The series converges absolutely for every x.

II. The series converges for x = c, and diverges for other x.

III. There exists

for x with |x – c| < R, i.e. for x with c – R < x < c + R, and diverges for | x – c | > R.

The series may or may not converge for x = c – R or x = c + R.

Proof : We assume, without loss of generality, that c = 0. Then the series becomes

Let R = sup

If R = 0, then the series converges only for x = 0.

If R > 0, then the series converges absolutely for every real x with | x | < R, because it converges for some real number x_0 with | x | < | x_0 |

Furthermore the definition of R implies that the series diverges for every real x with | x | > R.

If R =

Definition :

If power series

converges for | x – c | < R, and diverges for | x – c | > R, then is called the radius of convergence of the power series.

The set of all points where the power series converges in known as interval of convergence. It is onle of the following

]

Determination of radius of convergences, interval of convergence

We can computer the radius of convergence with the help of strengthened ratio that. Of course, it cannot be applied to every power series.

Theorem 6.3 Suppose that

exists or diverges to infinity. Then R is the radius of convergence of the power series.

.

Proof : Let

Then the power series converges if

diverges if

The following theorem, due to the French mathematician Jacques Saloma Hadamard (1865 – 1963), can be applied for computing the radius of convergence of a general power series.

Theorem 6.4 The radius of convergence R of the power series

is given by

where R = 0 if the lim sup diverges to , and R =

is 0.

Proof : Let

Then the power series converges if

and diverges if

Thus, the result is proved.



CHECK YOUR PROGRESS 2

Q 1: Find the radius of convergence, and the interval of convergence of each of the following series

(i)

(ii)

(iii)

6.5 UNIQUENESS OF POWER SERIES

Now, we consider the power series



has radius of convergence R > 0. Let



for all 

Then f is differentiable on] c – R, c + R [and for each



Uniqueness of Power Series representation

If 

and  $_n = b_n$, n = 0, 1, 2, 3.....

We have

 (1)

Also

 (2)

From (1), f (c) = a_0

From (2), f (c) = b_0

i.e., 

Since f is differentiable on] c – R, c + R[, we have



 (3)

Also



 (4)

From (3) and (4), we get



We note that

Now

Also

From (5) and (6), we get

Now f has derivatives of all orders on] c – R, c + R [with

and

and

Similarly, we will get

and =

and

Thus,

$a_n = b_n$, for n = 0, 1, 2, 3 .....

Note : f may have different representations at different centre points.

Do You Know?

Some important power series with their interval of convergence are given below.

1.

2.

3.

4.

5.

6.

7.

(i) If n is zero or a positive integer, the series or a positive.

(ii) If n > 0, but is not an integer, the series converges for

(iii) If –1 < n < 0, the series converges for

(iv) If , the series converges for

6.6 PROPERTIES OF POWER SERIES

Let us consider the power series and

with radius of convergence $R_1(>0)$, $R_2(>0)$ respectively.

Let f(x), and g(x) be the functions defined by the two power series respectively, defined for all x

I =] c – R, c+R [,

Where R = min $\{R_1, R_2\}$.

We note that both the power series converge for all x

We state the following properties.

I. Equality of power series

The two power series defined by f(x) and g(x) are equal for all x if and only if

$a_n = b_n$ for all n = 0, 1, 2, ...

In particular, if for all x

$a_n = 0$ for all n = 0, 1, 2, ...

II. Term by term addition

For all x

f(x) + g(x) = $_n+b_n) (x–c)^n$

III. Multiplication of power series

We define

$d_0 = a_0 b_0$, and inductivdy

Then for all x

f(x) g(x) = $_n(x–c)^n$

We note that for any n ^{n} in

is

IV. Term-by-term differentiation

The term-by-term differentiation of the power series defined by f(x) is given by

We note that the radius of convergence of this new series is also R_1.

V. Term-by-term integration

The term-by-term integration of the power series defined by f(x) is given by



We note that the radius of convergence of this new series is also R_1.

Example : Let 

I [, and 

the interval of convergence, and the radius interval of convergence, and the radius of convergence of the power series 

Solution : We have]–1, 1[

 the interval of convergence of the power series 

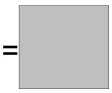
]–I, I[

radius of convergence of 

radius of convergence of 

 radius of convergence of 

Example : Using the power series representation f(x) =  $^2+x^3+$.

. . . , –1< x <1, find a power series representation for g(x) = 

x < 1

Hence evaluate the sum of the series 

Solution : We have



= 1 + x + x^2 + x^3 + . . .

= 

We have g(x) =

= $f'(x)$

=

= $^{2} + x^{3} + \ldots$

= $0 + 1 + 2x + 3x^{2} + \ldots$

= ^{n}, $-1 < x < 1$

Second Part

We have

Putting

i.e.



6.7 LET US SUM UP

- An infinite series of the form

  where

  c are real numbers, is called a power series about the point c. C is called the centre of the power series.

  Taking c = 0, we get the power series about 0, viz,

  It can be denoted by

- If the power series

  converges for $|x - c| < R$, and diverges for $|x - c| > R$, than

is called the radius of convergence of the power series.

The set of all points where the power series converges is known as interval of convergence. It is one of the following

$]c-R,\ c+R[$, $]c-R,\ c+R]$, $[c-R,\ c+R[$, $[c-R,\ c+R]$

- Let us assume that the power series  convergence $R > 0$.

  Let $f(x) =$ , for all 

  Then f is differentiable on $]c-R, c+R[$ and for each ,

6.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q1 (i) Here 







$= |x|$

 The series converges for $|x| < 1$, and diverges for $|x| > 1$.

When x = 1, the series becomes 

clearly, it is convergent (p-series, p = 2) when x = –1, the series becomes

The sequences

decreasing sequence, and

The series

Thus, the power series is convergent for

diverges for other real values of x.

(ii) Here

The series converges for , and diverges for

Now

When

When x = –2, the series is

Again , when x = 8,

when x = 8, the series is

The power series

and divergent for other real values of x.

Note : Here the region of convergence is] –2, 8 [, and its middle point 3 is the centre of the power series.

(iii) Here

The power series is convergent for all real x.

Check Your Progress 2

Ans to Q1 (i) Here

c – R = 0 – 1 = –1, c + R = 1

When x = 1, the series is convergent

when x = –1, the series is also convergent

radius of convergence = 1,

interval of convergence is [–1, 1]

(ii) Here



c – R = 3 – 5 = –2, c + R = 8

when x = –2, the series is divergent

when x = 8, the series is also divergent

radius of convergence = 5

interval of convergence in] –2, 8 [

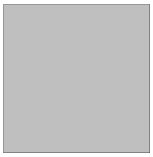
(iii) Here 





radius of convergence =

interval of convergence is] –, ,



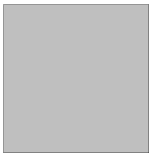
6.9 FURTHER READING

1) Principles of Mathematical Analysis by Walter Rudin

2) Methods of real Analysis by Richand R. Goldbery

3) Mathematical Analysis by S. C. Malik and Savita Arora

4) A Course of Mathematical Analysis by shanti Narayan.



6.10 MODEL QUESTIONS

Q 1: Find the radius of convergence, and the interval of convergence of each of the following series

(i) 

(ii) 

(iii) 

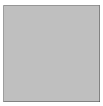
(iv) 

(v) 

(vi) 

(vii) 

Q 2: Let 

0). Construct a power series , other than 

the radius of convergence of the series 

[We take 

*** ***** ***

UNIT 7 : LIMIT OF FUNCTIONS

UNIT STRUCTURE

7.1 Learning Objectives

7.2 Introduction

7.3 Limit of a Function

7.4 Algebra of Limits

7.5 Right-hand and left-hand Limits

7.6 Infinite Limits

7.7 Let Us Sum Up

7.8 Answers to Check Your Progress

7.9 Further Reading

7.10 Possible Questions

7.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- limit of a function
- algebra of limits
- right hand and left hand limits
- infinite limit.

7.2 INTRODUCTION

In this unit we will give a precise definition of limit due to the German mathematician Karl Thlep Theodor Wilhelm Weierstrass (1815-1897). Though Isaac Newton (1643-1727), d'Alembert (1717-1783), Cauchy (1789-1857) had a clear idea about limits, the rigorous definition known as ε-δ definition, stated by Weierstrass around 1850. We will also study algebra of limits, and related theorems, right hand limit and left hand limit, infinite limit.

7.3 LIMIT OF A FUNCTION

Let f be a function defined at all points near c, except possibly at c.

Let L be a real number

f is said to tend to the limit L as x tends to (or approaches to) c, if for each , there exists a

i.e. when

i.e. , except possibly at c.

Symbolically, we write

Nota Bene

1. We have
   x = a.
2. We take any and then choose some i.e., The number is given first, and the number
3. For each , we can choose
   distance from c is less than
   be less then
4. The above definition does not give the value of L. It helps to examine whether a given number L is a limit or not.

Example 1 : Prove that

Solution : We note that

Now

, as

As x approaches 3, we can intuitively say that (x+3) approaches 6.

Now, we will prove that

Now f (x) – L = x+3 –6
= x – 3

which is in the form x – a, (

We take any

Now

We choose

Thus, we see that

Thus,

Example 2. Prove that

Solution : Let

We have

We choose

, when



CHECK YOUR PROGRESS 1

Q.1: Prove that

Q.2: Prove that

Q.3: Prove that

Theorem 7.1. Limit of a function, if exist, is unique.

Proof : Let

If possile, let f(x) tend to two limits L_1 and L_2 as x tends to c.

for any , we can choose

, when

, when

Now

, when

i.e. is less than any positive number

must be zero, i.e.

Thus, the limit is unique.

Theorem 7.2. (Sandwich Theorem)

Let f, g, h be three functions defined at on

possibly at c.

Let

(i)

(ii)

Then exists, and

Proof : We have

given , there exists

, when

Agian,

given , there exists

, when

Let

, when

i.e. , when

Also , when

i.e. , when

We have

, when

i.e. , when

i.e. , when

i.e.

7.4 ALGEBRA OF LIMITS

Let f and g be two functions defined on

g, fg,

, provided

Now we prove

Theorem 7.3. Let f, g be two functions defined at all points near c, except possibly at c. Let

Then (i)

(ii)

(iii) 

(iv) , provided 

Proof : (i) We have 

 given , there exists  such that 

when 

Again 

given , there exists 

Such that

, when 

Let 

, when 

and , when 

Now 

= 

i.e. , when 

i.e. 

(ii) Proceeding as above, we have





, for

i.e. , for

(iii) We have

given , there exist

, when for

Again,

given , there exist

, when for

Let

,

and , for

Now

,

Now

, for

We take

and

(1) becomes

=

= , for

i.e. , for

i.e.

(iv) Firstly, we prove that

we have

given , there exists

, for

Now

, for

i.e. , for

i.e. , for

We take



i.e. , for 

i.e. , for 

Again, 

 given , there exists 

, for 





, for 

(from 1 and 2)

, for 

i.e. , for 

i.e. 

Now 

= 

= 

Do You Know ?

I. Let f(x) = k, for all x,where k is a constant; i.e. f is a constant function. Then

, whatever may be the value

II. Let f(x) = x, for all x, i.e., f is an identity function. Then

.

, for choosing



CHECK YOUR PROGRESS 2

Q 1: If for all real x, find

Q 2: Prove that

Q 3: Prove that

Theorem 7.4. (Cauchy's Principle)

A necessary and sufficient condition for the existence of is that for a preassigned number exists , x_2 being any two numbers satisfying for ,

Proof : [We will only prove the necessary part].

Let

Then given , there exists

, for

Let x_1, x_2 be any two numbers satisfying

Then

=

i.e.

Thus the condition is necessary.



CHECK YOUR PROGRESS 3

Q 1: Prove that does not exsit

( )

Q 2: Prove that does not exist.

7.5 RIGHT-HAND AND LEFT-HAND LIMITS

In some cases, x approaches the number c form the right that is, through numbers bigger than c or from the left-that is, through numbers smaller than c. In some cases the function may be defined to the left or to the right of c. For example f(x) = /x is defined only for

Definition 1. (Right-hand limit)

A function f is said to tend to a limit L as x approaches c from the right if for each , there exists

, when

Symbolically, we write

or

Definition 2 (Left-hand limit)

A function f is said to tend to a limit L as x approaches c from the left if for each , there exists

, when

Symbolically, we write

or

Do You Know ?

[redacted] if and only if either [redacted] [redacted].

Theorem 7.4 Let f(x) be defined on an open interval containing c. Then [redacted] is defined if and only if [redacted] and [redacted] both defined and equal.

In this case

[redacted]

Proof : Let [redacted] exist, and let [redacted]

Then given [redacted], there exists [redacted]

[redacted], when [redacted]

Consequently [redacted]

Again, if [redacted], then [redacted]

Consequently [redacted]

Conversely, let [redacted]

[redacted]

We have [redacted]

[redacted] given [redacted], there exists [redacted]

[redacted], when [redacted]

Again, [redacted]

[redacted] given [redacted], there exists [redacted]

[redacted], when [redacted]

Let

Let

This implies either or

Now

Again

Thus



CHECK YOUR PROGRESS 4

Q 1: Evalute (i)

(ii) , where

Do You Know?

The greatest integer function (or the floor function), denoted [x], is defined as

[x] = the greatest integer less than or equal to x,

Thus [0.0005] = 0

[0.5] = 0, etc

It was first defined by the French mathematician Adrien-Marie Legendre (1752-1833] in 1798.

Example 3 Examine if

In $0 < x < 1$, $[x] = 0$

In $-1 < x < 0$, $[x] = -1$



Thus, 



Do You Know?

The sign function or signum function, denoted by sgn x, is defined by



Also $x = |x| \operatorname{sgn} x$,

$\operatorname{sgn} x^n = (\operatorname{sgn} x)^n$

, for 

Example 4 Examine if  exists

When $x > 0$, $\operatorname{sgn} x = 1$



When $x < 0$, $\operatorname{sgn} x = -1$



Thus 

 does not exist.

Example 5 Examine if 







Example 6 Examine if 









7.6 INFINITE LIMITS

Let us consider the function 

Clearly, it is a decreasing function. Also we note that f(x) comes closer and closer to zero as x becomes larger and larger. In this case we say that f(x) tends to zero as x tends to

Definition 1 A function f is said to tend to a limit L as x tends to

every , there is G > 0 such that , when x > G symbolically, we write 

In source cases, we are to examine the behaviour of a function f(x), as x takes smaller and smaller negative values.

Definition 2 A function f is said to tend to a limit L as x tends to –

every , there is G > 0 such that .

Symbolically, we write

Example 7 Prove that

Here

We take

Now

, whenever x > G, where

Example 8 Prove that

We have

Now we prove that

We take

Now

, whenever x > G, where

Now,

=



CHECK YOUR PROGRESS 5

Q 1: Prove that

Q 2: If –1 < x < 1, prove that (n is a positive integer)



7.7 LET US SUM UP

- Let f be a function defined at all points near c, except possibly at c.
  Let L be a real number.
  f is said to tend to the limit L as x tends to c, if for each
  exists a
- The
  whether a given number L is a limit or not.
- Limit of a function, if exists, is unique.
- Let f, g, h be three functions defined on
  possibly at c.
  Let
  (i)
  (ii)
  then, exists, and
  This theorem is known as Sandwich Theorem.
- A function f is said to tend to a limit L as x tends to
  , there is G > 0 such that
  , when x > G
  Symbolically, we write

7.8 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

Ans to Q No 1: We note that

Now

=

As x approaches 2, we can intnitively say that (x+2) approaches 4.

Now, we will proe that

Now f (x) – L = x + 2 – 4 = x – 2, which is in the form x – a, (a = 2)

We take any

Now

We choose

Thus, we see that

Thus,

Ans to Q No 2: Here

, when

i.e. , when

We choose

, when

i.e.

Ans to Q No 3: Here

We choose , and then

and

and $2 < x < 4$

and

and

and

Now, we choose such that , i.e.

Also

taking

, whenever

, whenever

Check Your Progress 2

Ans to Q No 1: (given)



Now 



by Sandwich Theorem, 

Ans to Q No 2: Here 

In 



 and 

by Sandwich Theorem, 

i.e. 

Again, in 



Now , and 

by Sandwich Theorem, 

i.e. 

Thus, 



Ans to Q No 3: Try Yourself

Check Your Progress 3

Ans to Q No 1: Try yourself

Ans to Q No 2: Try yourself

Check Your Progress 4

Ans to Q No 1: (i) We have

, whereas 



(ii) Try yourself

Check Your Progress 5

Ans to Q No 1: We have



Let 

Now , whenever 

i.e., whenever 

i.e. whenever 

i.e. , whenever x > G, where 

i.e. 



7.9 FURTHER READING

1) Fraleigh, John B : A first course in abstract Algebra, Pearson Education (Singapure) Pvt. Ltd. India Branch, Delhi - 11092.
2) Singh, S and Qazi, Z : Modern Algebra, Vikas Publicasting House Pvt. Ltd, New Delhi - 110014.
3) Macauley, M. : Section 6 : Fields and Galois Theory, http :// www.math.clemson.edu/~macaule/.
4) Talati, A. : Class Notes of Professor June Huh, 2022.
5) Singh, J. : Theory of Field Extension, Maharshi Dayanand University Press, Rohtal-124001.

7.10 MODEL QUESTIONS

Q 1: It is given that

Express it in

[There exists , such that, for all

and

Q 2: Evaluate

Q 3: If , show that

Q 4: Show that

Q 5: Evaluate 

Q 6: Prove that 

Q.7: Show that 

*** ***** ***

UNIT 8 : CONTINUOUS FUNCTIONS

UNIT STRUCTURE

8.1 Learning Objective

8.2 Introduction

8.3 Definition and Examples of Continuous Function

8.4 Types of Discontinuity with Examples

8.5 Algebra of Continuous Function

8.6 Properties of Continuous Function

8.7 Definition and example of Uniform Continuity

8.8 Let Us Sum Up

8.9 Answers to Check Your Progress

8.10 Further Reading

8.11 Model Questions

8.1 LEARNING OBJECTIVES

At the end of this unit, we will be able to

- define and learn about continuity of a function
- know examples of continuous functions
- know about the types of discontinuity
- know examples of discontinuous functions
- learn about the properties of continuous functions
- define and know about uniform continuity of a function.

8.2 INTRODUCTION

In the previous unit, we studied about the limit of functions. In this unit, we study about continuous functions which comprise of an important special class of functions in real analysis. The word “continuous” means “without interruption”. In case of functions, the word “continuity” holds the same meaning. Those function whose graph can be drawn without interruption (without holes or breaks) can be referred to as continuous function. In section, we discuss about the properties of continuous functions

which make it so important. We introduce the concept of uniform continuity in section.

8.3 DEFINITION AND EXAMPLE OF CONTINUOUS FUNCTION

Definition 1: Let $\mathbb{R}$. A function $\mathbb{R}$ is said to be continuous at *c*, if the following conditions are satisfied:

1. *f*(*c*) is defined
2. exists
3.

If *f* fails to be continuous at *c*, then *f* is said to be discontinuous at *c*.

Definition 2: Let $\mathbb{R}$ and let $\mathbb{R}$. If then the function *f* is said to be continuous on the set *B* if *f* is continuous at every point of *B*.

Example : Prove that the constant function on $\mathbb{R}$.

We first prove the continuity of the constant function at a point $\mathbb{R}$. We know that is defined. Also, constant function. So, from the definition 1 of continuity, *f* is continuous at for any $\mathbb{R}$. Henceforth, *f* is continuous on $\mathbb{R}$.

Example : Prove that the identity function continuous at $\mathbb{R}$.

We first prove the continuity of the identity function at a point $\mathbb{R}$. We know that is defined. Also, definition 1 of continuity, *f* is continuous at $x = c$ for any $\mathbb{R}$. Henceforth, *f* is continuous on $\mathbb{R}$.

Example : Prove that the modulus function is a continuous function on $\mathbb{R}$.

We can write $|x|$ as

It is clear that $x > 0$ and $x < 0$. So, to

prove the continuity on $\mathbb{R}$, we only need to prove the continuity of $x =$ 0. Since $x = 0$, we need to prove that know that and Since henceforth modulus function is continuous on $\mathbb{R}$.

In other words, we can define continuous function as follows :

Definition 3 : Let $\mathbb{R}$ and $\mathbb{R}$ be a function with domain A and co-domain $\mathbb{R}$. For any f is said to be continuous at c, if, given any number there exist a number whenever

Note: The definition of continuity is also known as the Cauchy's definition.

Example: Prove using $f(x) : = x$ is continuous at a point $x = 2$.

We need to show that for any there exists whenever

Here $f(2) = 2$. So

Now, for it is clear that whenever

Example: Using is a continuous function of $x = 3$.

We need to show that for any there exists whenever

Here, So .

So for

Now, taking it is clear that whenver

Example: Using definition of continuity, prove that is a continuous function at

We need to show that for any there exists whenever

Here, $f(3) = 9$. So

So for

Here .

Let us take Then

Now, taking it is clear that



CHECK YOUR PROGRESS 1

Q 1: Prove that is continuous on $\mathbb{R}: x > 0\}$

Q 2: Determine a condition on $|x-1|$ that will assure that

Q 3: Using definition of continuity, prove that is continuous at $x = 1$.

Q 4: Let and let $f : \mathbb{R}$ $\mathbb{R}$ satisfy the condition for all x, y $\mathbb{R}$. Show th f is continuous at every point $\mathbb{R}$.

Q 5: Prove that the function is continuous at .

8.4 TYPES OF DISCONTINUITY WITH EXAMPLES

In the previous section, we defined the continuity of a function at a point c. We see in Definition 1 that if one or more conditions of this definition fails to hold, then the function is discontinuous at point $x = c$. Based on the failure of one or more out of the three conditions in definition..... types of discontinuity that occurs are as follows :

1. **jump discontinuity** occurs when the right-hand limit and left-hand limit both exist but are not equal. Hence violating the second condition.
2. **infinite discontinuity** occurs when one-sided limits (left-hand limit and right hand limit) of the function is infinite. Thus not exist, violating the second condition.
3. **removable discontinuity** occurs when equal to $f(c)$. This violates the third condition of definition on continuity.

Note : Discontinuity occurs when the function is not defined at c. Based on the types of discontinuities we present few examples below.

Example : Show that the function 

$x = 3$. The given function  $x = 3$. The condition 1 of the definition of continuity is not satisfied and henceforth the function is discontinuous at $x = 3$.

Example : Show that the function  removable discontinuity at $x = 2$.

Here we see that 

But $f(2) = 3$. The $f(x)$ has removable discontinuity at $x = 2$.

Example : Verify the continuity of the function  at $x = 2$.

Here .

Hence the function is continuous at $x = 2$.

Note : The removable discontinuity in any function can be removed by..... as can be seen in the above examples.

Example : Show that the function  discontinuity at $x = 3$.

Here  but 

Hence  of definition 1. So, the function has jump discontinuity at $x = 3$.

Example : Show that the function  discontinuity at $x = 3$

Here 

It is clear that both left-hand limit and right-hand limit of $f(x)$

approaches [ ] x approaches 3. Hence, there exist infinite discontinuity at $x = 3$.



CHECK YOUR PROGRESS 2

Q 1: Verify the continuity of the signum function at $x = 0$.

Q 2: Determine the points of continuity of the greatest integer function [ ]

Q 3: Show that [ ] is not continuous at $x = 0$.

Q 4: Show that the function [ ] is not continuous at $x = 0$ but [ ] is continuous at $x = 0$.

8.5 ALGEBRA OF CONTINUOUS FUNCTION

In this section, we look at the algebra of continuous functions for the four arithmetic operations viz. addition, subtraction, multiplication and division.

Theorem : Let [ ] $\mathbb{R}$ and f and g be functions from A to $\mathbb{R}$ which are continuous at $x = c$. Then

(a) $f + g$ is continuous at c

(b) $f - g$ is continuous at c

(c) fg is continuous at c

(d) bf is continuous at c

(e) bg is continuous at c

(f) f/g is continuous at c if [ ] c if $g(c) = 0$.

Proof : (a) Since f and g are continuous at c, so by definition of continuity we have

[ ] and [ ]

But we know from properties of limit that

[ ]

Therefore $f + g$ is continuous by definition 1 of continuity.

(b) Similar as above, we have from properties of limit

[ ]

Therefore .

(c) In similar fashion, from the properties of limit we have

Therefore *fg* is continuous by definition 1 of continuity.

(d) Let

Hence, *f/g* is continuous at *c* for

For the case

condition of definition 1 of continuity, the function *f/g* is discontinuous at *c*.

Example : Prove that *n*.

From the example on the previous section, we know that .

Hence, from the algebra of continuous function, we have

Continuing as above, we get

Henceforth,

Example : Prove that polynomial functions are continuous on $\mathbb{R}$.

A polynomial function is of the form

for all $\mathbb{R}$, where s.

For any point $x = c$, $p(c)$ is defined and given as

From the algebra of continuous functions, we have

Hence polynomial function are continuous on $\mathbb{R}$

Next theorem shows that the composition of two continuous function is continuous.

Theorem : For *A, B* $\mathbb{R}$, let $\mathbb{R}$ be a continuous functions at and for let $\mathbb{R}$ be a continuous functions Then the composition $\mathbb{R}$ is continuous at *c*.

Proof : Since *g* is continuous function at *b* = *f*(*c*), so for we have

whenever

or we can write

whenever

But for any there exists

whenever

Henceforth, for any there exists

whenever

Hence the result.

Theorem : For $\mathbb{R}$, let $\mathbb{R}$ be a continuous functions on *A* and for let $\mathbb{R}$ be a continuous functions on B. Then the composition $\mathbb{R}$ is continuous on *A*.

Proof : The theorem follows from the preceding result.

8.6 PROPERTIES OF CONTINUOUS FUNCTION

At first we define the Sequential criterion of continuity and later move to discuss about the functions continuous on intervals.

Sequential criterion for continuity : A function $\mathbb{R}$ is continuous at *x* = *c* if and only if for every sequence in *A* that converges to *c*, the sequence *f*(*c*)

Proof : Let as , i.e. for any there exists such that

whenever

Again, let *c*. So there exists a natural number *m* such that

for

Hence for

i.e. the sequence *f*(*c*).

Conversely, let for every sequence $\quad$ A that converges to c, the sequence $\quad$ $f(c)$.

Suppose that the function f has no limit i.e. there exists at least one $\quad$ for which $\quad$ does not exist. This means there is $\quad$ $\quad$ implies $\quad$

Thus for successive values of $\quad$ there is value $\quad$ $\quad$ and $\quad$. This implies that $\quad$ c but $\quad$ does not tend to $\quad$ supposition that limit does not exist is disproved.

Functions that are continuous on intervals have many important properties which are not possessed by general continuous functions. We establish some of the important results here in this section. We start by the general definition of a bounded function.

Definition : A function $\quad \mathbb{R}$ is said to be bounded on A if there exists a constant $\quad$ such that $\quad$ for all $\quad$

Note: Continuous functions need not be bounded.

Example: The function $\quad \mathbb{R}$ such that $\quad$ continuous on $\quad$. But f is not bounded on $\quad$ because for any $M > 0$, there exist point $\quad$ in $\quad$ such that $\quad$

We now state an important theorem based on the boundedness of a continuous function.

Boundedness Theorem : Let $\quad \mathbb{R}$ be continuous on closed bounded interval $\quad$ f is bounded on I.

Proof : We show contra-positively, that if the function is not bounded on I, then it fails to be continuous at some point in the closed interval.

Let if possible f is not bounded above.

So for any $\quad \mathbb{N}$, there exist a number $\quad$ such that $\quad$

Now $\quad$ I is bounded and has at least one limit point, say $\quad$ where $\quad$

Further, since $\quad$ is a limit point of the sequence $\quad$ there exists a sub-sequence $\quad$ of $\quad$ that converges to $\quad$ as $\quad$

Since I is closed and the elements of $\quad$ I, so $\quad$

Then, f is continuous at $\quad$. So $\quad$ converges to $\quad$

Thus the convergent sequence must be bounded.

But for all $\mathbb{N}$, which is a contradiction.

Thus the result follows.

Theorem : Let $\mathbb{R}$ be continuous function on *f* attains its bounds at least once in [*a, b*]

Proof : If *f* is a constant function, then trivially it attains its bounds at every point of the interval.

Let *f* be a function which is not a constant.

Since *f* is continuous on the closed interval [*a, b*], therefore, it is bounded. Let *m* and *M* be the infimum and supremum of *f*.

We need to showo that there exists point *a, b*] such that

On contrary, let us suppose *f* does not attain the supremum *M* i.e. for any .

We consider the function for all which is positive for all values of Thus the function *g* is continuous and so bounded in [*a, b*]. Let *k* (> 0) be its supremum.

Then,

which contradicts the hypothesis that *M* is the supremum of *f* in [*a, b*].

Thus our supposition that *f* does not attain the value *M* leads to a contradiction and therefore *f* attains its supremum for at least one value in [*a, b*].

It may similarly be shown that the function also attains its infimum *m*. Hence, the function attains its bounds at least once in [*a, b*].

Location of roots theorem : Let $\mathbb{R}$ be continuous on closed bounded interval *I* : = [*a, b*]. If or if there exists a number such that

Proof : Let us suppose

of intervals by successive bisections. Let where Let be the midpoint. If we take we prove the theorem. If then either or

If then we set and

If then we set and

In either case, we let

Thus we have and

We continue the bisection process. Suppose that the intervals have been obtained by successive bisection in the same manner. Then we have and we set

If we take and we prove the theorem. If then either or

If then we set and

If then we set and

In either case, we let such that .

If the process stops by locating a point such that we are done.

If the process does not stop, then we obtain a nested sequence of closed bounded intervals such that for every $\mathbb{N}$, we have and

Furthermore, since the intervals are obtained by repeated bisections, the length of is property that there exists a point for all $\mathbb{N}$. Since $\mathbb{N}$, we have .

Hence it follows that

Since *f* is continuous at *c*, we have

Also since for all $\mathbb{N}$, thus

Also since for all $\mathbb{N}$, thus

Thus we conclude that $f(c) = 0$ and *c* is a root of *f*.

Intermediate Value Theorem : If *f* is continuous on a closed interval [*a, b*] and *k* is any number between *f*(*a*) and *f*(*b*), then there exists at least a point such that

Proof: Suppose that and let By the Location of root theorem, there exists a point *c* with such that Thus

If and let Then of root theorem, there exists a point *c* with Thus

8.7 DEFINITION AND EXAMPLE OF UNIFORM CONTINUITY

Definition: Let $\mathbb{R}$ and let $\mathbb{R}$ be a function whose domain is *A* and there exists a such that for any numbers whenever

Note: From the definition, it is clear that if *f* is uniformly continuous on *A*, then it is continuous at every point in *A*. However, the converse is not true.

Theorem: Non-uniform Continuity Criteria Let $\mathbb{R}$ and let $\mathbb{R}$. Then the following statements are equivalent:

i. *f* is not uniformly continuous on *A*

ii. There exists an such that for every points such that and

iii. There exists an and two sequences and *A* such that and for all $\mathbb{N}$

Theorem: A function which is continuous on a closed interval is also uniformly continuous on that interval.

Proof: If *f* is not uniformly continuous on *I*, then by the non-uniform continuity criteria, there exists an and two sequences and in *I* such that and for all $\mathbb{N}$.

Since *I* is bounded, the sequence s a sub-sequence of *z*. Since *I* is

closed, the limit z belongs to I.

Also the corresponding sub-sequence z since

Now if f is continuous at the point z, then both sequences and $f(z)$. But it is not possible since

for all $\mathbb{N}$. Thus our assumption that f is not uniformly continuous on I implies f is not continuous at some point positivity, f is continuous at every point of I implies f is uniformly continuous on I.



8.8 LET US SUM UP

- Let $\mathbb{R}$. A function $\mathbb{R}$ is said to be continuous at c, if the following conditions are satisfied:
  1. $f(c)$ is defined
  2. exists
  3.
- Let $\mathbb{R}$ and let $\mathbb{R}$. If f is said to be continuous on the set B if f is continuous at every point of B.
- Let $\mathbb{R}$, and $\mathbb{R}$ be a function with domain A and co-domain $\mathbb{R}$. For any f is said to be continuous at c, if, given any number there exist a number whenever
- **Jump discontinuity** occurs when the right-hand limit and left-hand limit both exist but are not equal. Hence violating the second condition.
- **Infinite discontinuity** occurs when one-sided limits (left-hand limit and right hand limit) of the function is infinite. Thus not exist, violating the second condition.
- **Removable discontinuity** occurs when

equal to $f(c)$. This violates the third condition of definition on continuity.

- Let $\mathbb{R}$ and let f and g be functions from A to $\mathbb{R}$ which are continuous at $x = c$. Then
  (a) $f + g$ continuous at c
  (b) $f - g$ is continuous at c
  (c) fg is continuous at c
  (d) bf is continuous at c
  (e) bg is continuous at c
  (f) f/g is continuous at c if c if $g(c) = 0$.
- For $\mathbb{R}$, let $\mathbb{R}$ be a continuous functions at let $\mathbb{R}$ be a continuous functions at Then the composition $\mathbb{R}$ is continuous at c.
- For $\mathbb{R}$, let $\mathbb{R}$ be a continuous functions on A and for and let $\mathbb{R}$ be a continuous functions on B. Then the composition $\mathbb{R}$ is continuous on A.
- A function $\mathbb{R}$ is continuous at $x = c$ if and only if for every sequence A that converges to c, the sequence converges to $f(c)$.
- Let $\mathbb{R}$ be continuous on closed bounded interval Then f is bounded on I.
- Let $\mathbb{R}$ be continuous function on f attains its bounds at least once in $[a, b]$
- Let $\mathbb{R}$ be continuous on closed bounded interval or if such that $f(c) = 0$
- If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there exists at least a point that
- Let $\mathbb{R}$ and let $\mathbb{R}$ be a function whose domain is A and co-domain is $\mathbb{R}$. The function f is said to be uniformly continuous on A if for each there exists a whenever

- If *f* is uniformly continuous on *A*, then it is continuous at every point in *A*. However, the converse is not true.
- Let $\mathbb{R}$ and let $\mathbb{R}$. Then the following statements are equivalent :
  - i. *f* is not uniformly continuous on *A*
  - ii. There exists an such that for every points such that and
  - iii. There exists an and two sequences and and *A* such that and for all $\mathbb{N}$
- A function which is continuous on a closed interval is also uniformly continuous on that interval.



8.9 ANSWERS TO CHECK YOUR PROGRESS

Answer to Check your progress 1

Ans to Q No 1: We have to prove is continuous on $\mathbb{R}$:

x > 0} i.e. we need to prove that for there exists

whenever

i.e. whenever

But

for

So we need to have an upper bound for the term 1/(*cx*) that holds in some neighbourhood of *c*. Let

which implies

So we have for

Thus we can write 

Now to make the term on the right hand side less than 



Thus choosing  we get

 whenever 

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

Ans to Q No 4: Try yourself.

Ans to Q No 5: Try yourself.

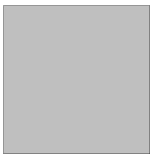
Answer to Check your progress 2

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

Ans to Q No 4: Try yourself.

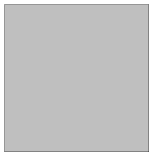


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8.11 MODEL QUESTIONS

Q 1: Show that

Q 2: Show that is not uniformly continuous on

Q 3: Show that the function

Q 4: Prove that $x = 2$

Q 5: Show that the function $x = 0$

*** ***** ***

UNIT 9 : DIFFERENTIABLE FUNCTIONS

UNIT STRUCTURE

9.1 Learning Objective

9.2 Introduction

9.3 Definition of Derivative

9.4 Algebra of Derivatives

9.5 Darboux's Theorem

9.6 Rolle's Theorem

9.7 The Mean Value Theorem

9.8 Raylor's Theorem

9.9 Taylor's Series and Maclaurin's Series

9.10 Let Us Sum Up

9.11 Answers to Check Your Progress

9.12 Further Reading

9.13 Model Questions

9.1 LEARNING OBJECTIVES

At the end of this unit, we will be able to :

- learn about derivative of a function and the algebra of derivatives
- know about Rolle's theorem
- know about Lagrange's mean value theorem
- know about Cauchy's mean value theorem
- learn about the Taylor's theorem with Lagrange's form of remainder
- learn about the Taylor's theorem with Cauchy's form of remainder
- learn about Taylor's series and Maclaurin's series.

9.2 INTRODUCTION

The geometrical and physical interpretation of the derivative of functions is familiar to us due to its vast applications in physics, economics, etc. In this unit, we discuss about the theory of differentiation by concentrating solely on the mathematical aspects of the derivative. In this unit, we define

the derivative of a function and discuss about the properties of derivatives. In the succeeding section, two important theorems are discussed viz. Lagrange's Mean value theorem and Cauchy's Mean value theorem. Its applications are also discussed in the section. In the last section, Taylor's theorem is discussed.

9.3 DEFINITION OF DERIVATIVES

Definition 1: Let R be an interval, and

with domain *I* and co-domain R. For any *L* is said to be the derivative of *f* at *c*, if, given any number such that for any

whenever

In this case, *f* is said to be differentiable at *c* and we denote for *L*.

In other words, we define derivative of *f* at *c* as follows :

Definition 2 : Let R be an interval, then the function is said to be derivable at *c* if

exists.

In this case, we write

Note: is called the left-hand derivative of *f* at *c* and *f* at *c*. Thus the derivative exists when left-hand derivative and right-hand derivative of *f* at *c* exists and are both equal.

Example : If for R, then for any R, find

We have for any R

Thus is defined for all points in R and for

The following theorem establishes that the continuity of *f* at a point *c* is a necessary condition (but not sufficient condition) for the existence of

the derivative at c.

Theorem : If R has a derivative at f is continuous at c

Proof : Let f has a derivative at c.

So

Now, we can write

Thus the function f is continuous at $x = c$.

Note: The continuity of f at a point does not assure the existence of the derivative at that point.

Example: The function $x = 0$ but not differentiable at $x = 0$.

Since

so $f(x)$ is continuous at $x = 0$.
But we have,

and

Thus

Hence the function is not derivable at $x = 0$.



CHECK YOUR PROGRESS 1

Q 1: Show that the function f on R defined by

Q 2: Show that the function for all R is continuous but not derivable at $x = a$

Q 3: Show that the function f defined as

is derivable at $x = 0$. Also prove that

9.4 ALGEBRA OF DERIVATIVES

In this section, we look at the basic algebra of derivatives which are very useful in the calculation of derivatives of various combinations of functions.

Theorem : Let R and

at c, where $I := [a, b]$ and

(a) For any R, the function c and

(b) The function is differentiable at c and

(c) The function fg is differentiable at c and

(d) For f/g is differentiable at c and

Proof : (a) We have

Hence for any R, the function is differentiable at c and

(b) We have

Hence function *c* and

(c) We have

Hence function *fg* is differentiable at *c* and

(d) Let *g* is differentiable at *c*, so it is continuous at *c* i.e. Since (given) implies either or Thus if there exists an interval with such that , if then there exists an interval with that . In either case we get

So we have,

Hence for *f/g* is differentiable at *c* and

Corollary : If *I* to R that are differentiable at

1. the function *c* and
2. the function *c* and
3. for

Theorem : Caratheodory's Theorem: Let Then *f* is differentiable at on *I* that is continuous at *c* and satisfies for In this case,

Proof : Let us suppose *f* is differentiable at *c* i.e.

We defined

Clearly is continuous function and

If $x = c$, then the result follows.

If we multiply by $f(x) - f(c) =$

Conversely, suppose there exist *c* and

Now for



But



i.e. *f* is differentiable at *c* and

Now we establish the theorem on differentiation of composite functions, also known as Chain rule, as follows :

Chain Rule: For the intervals *I, J* in R, let R and be such that *f* is differentiable at *g* is differentiable at *f*(*c*), then the composite function *gof* is differentiable at *c* and

Proof: Since there exists function on *J* such that *c* and for all and

Also, since exists, so by the above Caratheodory's theorem, there exists function *I* such that *f*(*c*) and for all and

Thus substituting and we get

for all such that

Again since the function *c* and its value at *c* is

Theorem: If *f* is derivable at *c* and *f* is also derivable at *c* and

Proof : We have

where for

by the Chain rule

since

9.5 DARBOUX'S THEOREM

In the succeeding sections, we establish some important theorems related with the derivatives of function. We start the section by definition of increasing and decreasing functions:

Definition: A function *f* is said to be increasing function in the interval [*a, b*] if

Definition: A function *f* is said to be decreasing function in the interval [*a, b*] if

Remark: The function *f* is said to be strictly increasing or strictly decreasing in [*a, b*] if

Definition: A function is said to be monotonic in an interval [*a, b*] if it is either increasing or decreasing in the interval.

Let us suppose the function *f* is derivable at $x = c$ i.e.

Thus for any there exist

whenever

i.e. whenever

Now, for we obtain

whenever

Thus we have for

i. for

ii. for

i.e. the function is increasing at *c* if

decreasing at *c* if

Note: At theend point *a*, there exist an interval

i. for

ii. for

Note: At the end point *b*, there exist an interval

i. for

ii. for

We now state the Darboux's theorem as follows :

Darboux's Theorem : If a function *f* is derivable on a closed interval [*a, b*] and

point *c* between *a* and *b* such that

Proof: Without loss of generality, let us assume .

Since , so there exists a number

(1)

Also since so there exists

(2)

Also since *f* is derivable in [*a, b*], it is continuous in [*a, b*].

But we know from the previous chapter, that continuous functions in closed interval are bounded and attains its bounds. So there exists a point [*a, b*] such that *m* is the infimum of the function *f*.

It is clear from (1) and (2), that *c* cannot coincide with *a* or *b* since otherwise there would exist point where the values of the function would be less than *f*(*c*). Thus *c* is an interior point of [*a, b*].

Now we prove that

If then there exists an interval

for all *m* is the infimum of *f*.

Similarly, if then there exists an interval

such that for all which is again a contradiction.

Hence the only possibility is

9.6 ROLLE'S THEOREM

In this section, we establish the Rolle's Theorem as follows :

Rolle's Theorem : If a function *f* is (i) continuous on a closed interval

[*a, b*], derivative *a, b*) and (iii)
there exists at least one point *c* in (*a, b*) such that

Proof: Since the function is continuous on a closed interval [*a, b*], it is bounded and attains its bounds i.e. for infimum *m* and supremum *M* of the function *f*, there exists points *c* and *d* in [*a, b*] such that

and

There are two possibilities: either $m = M$ or

If $m = M$, then *f* is constant function and therefore its derivatives

If *m* is different from

i.e.

This means

We now show that

If then tere exists an interval

point *x* of which, *m* is the infimum of *f*.

Similarly, if then there exists an interval

for every point *x* of which,

Hence the only possibility is

9.7 THE MEAN VALUE THEOREM

In this section, we discuss about Mean Value theorem and some of its applications. The Mean Value theorem relates the values of a function to the values of its derivatives. At first we state the first Mean Value theorem viz. Lagrange's Mean Value Theorem and later state the second Mean Value Theorem viz. Cauchy's Mean Value theorem.

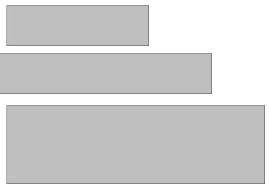
Lagrange's Mean Value Theorem : If *f* is continuous on a closed interval [*a, b*] and *f* has a derivative in (*a, b*), then there exists at least one point *c* in (*a, b*) such that

Proof: We consider a function



where A is a constant to be determined such that 

Therefore



Also, since the function 

function, so 

(i) continuous on [a, b]

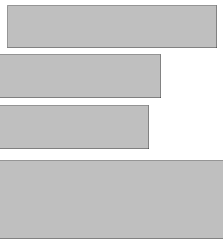
(ii) derivable on [a, b]

(iii) 

Thus, by Rolle's theorem, there exists a real number 

.

But



Thus

Or 

Cauchy's Mean Value Theorem : Let f and g be two functions on [a, b], such that, they are (i) continuous on [a, b], (ii) derivable on (a, b), and (iii)   Then there exists at least one point 

that



Proof: If  g being continuous and derivable on [a, b], we would get  contradiction.

Thus, we note that 

Now, consider the function



where A is a constant to be determined such that 



Also, since the function  is a sum of two continuous and derivable function, so 

(i) continuous on $[a, b]$

(ii) derivable on $[a, b]$

(iii) 

Thus, by Rolle's theorem, there exists a real number  that 

But



Thus 





Example: Prove that

 where 

Here  and  for 

Also,  and 

Since f and g are both continuous and differentiable, therefore by Cauchy's Mean Value Theorem on 







CHECK YOUR PROGRESS 2

Q 1: Prove that

if Also deduce that

9.8 TAYLOR'S THEOREM

In the previous sections, we have defined the derivative of the function f at $x = c$ and denoted it as

derivative of the function $x = c$ (if exists) and denote it as ,

the existence of implies existence and continuity of

neighbourhood of c. The higher order derivatives can be similarly defined and the derivative of c (if exists) is said to be the n^{th} derivative of f at c and is denoted as

We state the Taylor's Theorem as follows :

Taylor's Theorem : If a function f defined on $[a, a + h]$, is such that

(i) the derivative $a, a + h]$ and

(ii) the n^{th} derivative $a, a + h)$,

then there exists at least one real number

where p is a given positive number.

Proof: Condition (i) in the theorem implies that all derivatives exists and are continuous on $[a, a + h]$.

We consider the function $a, a + h]$ as

where A is a constant to be determined such that

Now

(4)

We observe that,

(i) the function $a, a + h]$ since are all continuous on $[a, a + h]$

(ii) the function $a, a + h)$ since for all r are derivable in $(a, a + h)$, and

(iii)

Thus the function satisfies all the conditions of Rolle's Theorem and hence there exists at least one number But

(5)

Substituting the value of A from (5) in (4), we get the required result,

Remark : The term

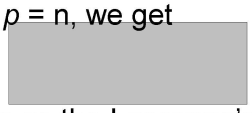
which occurs after n terms, is known as Taylor's remainder after n terms.

Remark : For $p = 1$, we get



which is known as the Cauchy's form of remainder.

Remark : For p = n, we get



which is known as the Lagrange's form of remainder.

Second form of Taylor's Theorem : Substituting  or  (3), we get





where 

The remainder term after n terms can be written as



where c depends on x and $a < c < x$.

9.9 TAYLOR'S SERIES AND MACLAURIN'S SERIES

In the previous section, we observe from the equation (3) that the value $f(a + h)$ of the function at a point is approximated by a summation given in R.H.S of the equation. The remainder term R_n is used to estimate the error in approximating the function $f(a + h)$. However, a natural question arises as to how far the R.H.S of (3) correctly represents the L.H.S of (3). The question can be answered if the series



converges to $f(a + h)$.

Let 



Thus if  as 



i.e. the infinite series

converses to

Thus for a function f having derivatives of n^{th} order in $[a, a + h]$, if Taylor's Remainder as then

Definition: The infinite series

(7)

is said to be the Taylor's series.

Replacing

(8)

which is the expansion of $f(x)$ in powers of $(x - a)$.

Definition: For all we get from (8), (by replacing $a = 0$),

which is known as the Maclaurin's infinite series expansion of $f(x)$ in powers of x.

Example: Derive the expansion of

Here x.

Taking Lagrange's form of remainder, we get

Thus

i.e. as

Also $f(0) = 1$ and n.

Thus substituting these values in Maclaurin's series, we get

R



CHECK YOUR PROGRESS 3

Q 1: Find the expansion of sin x using Maclaurin's series.

Q 2: Find the expansion of cos x using Maclaurin's series.

Q 3: Find the expansion of log $(x + 1)$ using Maclaurin's series.



9.10 LET US SUM UP

- Let R be an interval, and R be a function with domain I and co-domain R. For any L is said to be the derivative of f at c, if, given any number there exist a number such that for any

  whenever

  In this case, f is said to be differentiable at c and we denote L.
- Let R be an interval, then the function derivable at c if

  exists.

  In this case, we write
- is called the left-hand derivative of f at c and is called the right-hand derivative of f at c. Thus the derivative

  derivative of f at c exists and are both equal.
- If R has a derivative at f is continuous at c.
- The continuity of f at a point does not assure the existence of the derivative at that point.

- Let R and *c*,
  where and
  (a) For any R, the function of *c* and
  (b) The function *c* and
  (c) The function *fg* is differentiable at *c* and
  (d) For *f/g* is differentiable at *c* and
- If *I* to R that are differentiable at
  (a) the function *c* and
  (b) the function *c* and
  (c) for
- Let R such that *f* is differentiable at and only if there exists a function *I* that is continuous at *c* and satisfies for In this case,
- For the intervals *I, J* in R, let R and If *f* is differentiable at *g* is differentiable at *f*(*c*), then the composite function *gof* is differentiable at *c* and
- If *f* is derivable at *c* and *f* is also derivable at *c* and
- Darboux's Theorem: If a function *f* is derivable on a closed interval [*a, b*] and one point *c* between *a* and *b* such that

- Rolle's Theorem: If a function f is (i) continuous on a closed interval [a, b], derivative a, b) and (iii) $f(a) = f(b)$, then there exists at least one point c in (a, b) such that
- Lagrange's Mean Value Theorem : If f is continuous on a closed interval [a, b] and f has a derivative in (a, b), then there exists at least one point c in (a, b) such that



- Cauchy's Mean Value Theorem: Let f and g be two functions on [a, b], such that, they are (i) continuous on [a, b), (ii) derivable on (a, b), and (iii) a, b). Then there exists at least one point such that



- Taylor's Theorem: If a function f defined on the derivative a, $a + h$] and (ii) the derivative a, $a + h$), then there exists at least one real number





where p is a given positive number.

- The term



which occurs after n terms, is known as Taylor's remainder after n terms.

- For $p = 1$, we get



which is known as the Cauchy's form of remainder.

- Foro $p = n$, we get



which is known as the Lagrange's form of remainder.

- Substituting or

  where *n* terms can be written as

  where *c* depends on *x* and $a < c < x$.

- The infinite series

  (9)

  is said to be the Taylor's series.

- For all

  is known as the Maclaurin's infinite series expansion of $f(x)$ in powers of *x* which is obtained by replacing $a = 0$ in the Taylor's series.

9.11 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1

Ans to Q No 1: At $x = 1$, we have

and

Since right hand derivative left hand derivative, so exist.

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

Check your progress 2

Ans to Q No 1: Try yourself.

Check your progress 3

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

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9.13 MODEL QUESTIONS

Q 1: Show that the function for all but not derivable at $x = 0$ and $x = 1$.

Q 2: Let f be a twice differentiable function such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove that there is at least one between a and b for which

Q 3: Find the expansion of using Maclaurin's series, for all where m is any integer.

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UNIT 10 : FUNCTIONS OF SEVERAL VARIABLES

UNIT STRUCTURE

10.1 Learning Objectives

10.2 Introduction

10.3 Structure of R^n

10.4 Explicit and implicit Functions

10.5 Limit and Continuity of Function from R^n to R

10.6 Directional Derivatives

10.7 Partial Derivatives

10.8 Let Us Sum Up

10.9 Answers to Check Your Progress

10.10 Further Reading

10.11 Model Questions

10.1 LEARNING OBJECTIVES

At the end of this unit, we will be able to :

- know about the structure of R^n
- learn about the limit and continuity of functions from R^n to R
- learn about the directional derivatives and partial derivatives.

10.2 INTRODUCTION

In the previous units, we studied about functions which depend on a single variable *x*. But in real-world phenomenon, a quantity being investigated usually depends on two or more independent variables. So we need to extend the basic ideas of functions of a single variable to functions of several variables. The derivatives of functions of several variables are more varied and more interesting because of the different ways in which the variables can interact. There is a vast application of these functions of several variables in probability theory, statistics, fluid dynamics, etc.

10.3 STRUCTURE OF R^n

From the previous units, we have observed that the real number system R is more than just a collection of points. It has various algebraic, and order structures. The same is true of many other important sets of mathematical objects. Among the most important are the euclidean set R^n which we study in this chapter. The collection of elements of the set R^n is in the form of *n*-tuples of real numbers which we define as follows :

Definition: The set R^n is defined as the set containing n-tuples of real numbers. That is, R^n is the Cartesian product of R given as :

$R^n = R \times R \times \ldots$ *n* times)

The element ^{n} are called the points of R^n. They can also be viewed as vectors, and R as a field of scalars. We define addition of vectors in R^n as coordinate-wise addition:

If and

(1)

For R and ^{n} we define

(2)

We refer to such multiplication of a vector by a scalar as scalar multiplication. With these notions of addition and scalar multiplication, R^n possesses the algebraic structure of a vector space.

Theorem 1 : If addition and scalar multiplication on R^n are defined by (1) and (2), respectively, then the following are true for all ^{n} for all R:

(a) and R^n (closure of R^n under addition and scalar multiplication)

(b) (commutative law)

(c) (associative law)

(d) where 0 = (0, 0, ..., 0)

(e) and (distributive law)

(f)

(g)

(h)

We define subtraction as

then We can also write *x* and *x/a* for (1/*a*)*x*.

The Dot Product: Dot product of two vectors is defined as

(3)

where and

product (3) is a scalar. For that reason *x.y* is sometimes called the scalar product of *x* and *y*. It is also sometimes called the inner product of *x* and *y*. The dot product satisfies certain conditions that we summarize in next theorem.

Theorem 2 : If ^{n} and

(a) (commutative law)

(b) (distributive law)

(c)

Observe that *x.y* can equal 0 without either *x* and *y* being 0 [E.g., in R^2, (1, 1). (1, –1) = 1 –1 = 0].

Two nonzero vectors *x* and *y* for which *x.y* = 0 are said to be orthogonal vectors. Geometrically, two orthogonal vectors in R^n are perpendicular.



CHECK YOUR PROGRESS 1

Q 1: For *x* = (1, 2, 3) and *y* = (2, 3, 4), find *x.y*

Q 2: Prove that *x.x* = 0 if and only if *x* = 0

Q 3: For in let

(a) Show that "." satisfies the conditions of Theorem 2.

10.4 EXPLICIT AND IMPLICIT FUNCTIONS

If we consider a set of *n* independent variables *x, y, z*, ..., *t* and one dependent variable u, the equation

denotes the functional relation. In this case if assigned values of the independent variables, the corresponding values of the dependent variable *u* are determined by theh function relation.

The function represented by equation (4) is an explicit function but where several variables are concerned it is rarely possible to obtain an equation expressing one of the variables explicitly in terms of the others. Thus most of the functions of more than one variable are implicit functions, that is to say we are given a functional relation.

(5)

connecting the *n* variables *x, y, z, ..., t* and is not in general possible to solve this equation to find an explicit function which expresses one of these variables say *x*, in terms of the others *n* – 1 variables.

In this unit we shall be mainly concerned with the explicit functions.

An Explicit Function of Two Variables : If *x, y* are two independent variables and a variable *z* depends for its values on the values of *x, y* by a functional relation

(6)

then we say *z* is a function *x, y*. The ordered pair of numbers *x, y* is called a point and the aggregate of the pairs of numbers (*x, y*) is said to be the domain (or region) of definition of the function.

When the domain of definition is bounded by a closed curve *C*, it is said to be closed if *f* is defined for all points within and on the curve *C*; but open or unclosed when the function is defined for points within but not on the curve *C*.

The neighbourhood of a point : The set of values than *a, b* that satisfy the conditions

where

neighbourhood of the point (*a, b*).

Thus a neighbourhood is the square

where *x* takes any value from to *a*, and *y* from to *b*.

This is not the only way of specifying a neighbourhoodo of a point. There can be many other, though equivalent ways; for example the points inside the circle

(0, 0).

10.5 LIMIT AND CONTINUITY OF FUNCTION FROM R^N TO R

Definition : The limit of the function ^{n} R, with

R^n is the number R, denoted as

every there exists such that

(7)

Remarks :

(a) In Cartesian coordinates Then

is the distance between points in R^n,

(b) is the absolute value of real numbers.

The limit of functions *f :* R^2 R **:** A function *f* is said to tend to a limit *L* as a point (*x, y*) tends to the point if for every s such that

(8)

equivalently,

(9)

In other words, a function *f* tends to a limit *L*, when tends to the point if for every positive number

N of

for every point (*x, y*) other than *N*.

Symbolically we then write

(10)

L is the limit (the double limit or the simultaneous limit) of *f* when *x, y* tend to *a* & *b* simultaneously.

Computing a limit by the definition:

Example 1 : Use the definition of limit to compute

Solution: The function $f(x, y)$

First : Guess what the limit L is.

Along the line $x = 0$ the function is

Therefore, if L exists, it must be $L = 0$. Given find
Fix any number Given that find a number

i.e.,

Recall : that is,

Choose then

We conclude that $L = 0$.

Example 2 : Show that

Solution: Taking and

if



or if 

Thus for 

 when 



10. Algebra of Limits

In this section, we look at the algebra of limits for the four arithmetic operations viz. addition, subtraction, multiplication and division.

Theorem 3 : If *f, g* be two functions defined on some neighbourhood of a point  such that  and 

 then

(a) 

(b) 

(c) 

(d)  provided  when 

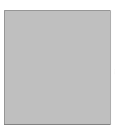
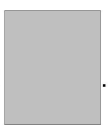
The proofs are exactly similar to those of the corresponding theorems for a single variable.

Remarks: The definition implies that there must be no assumption of any relation between the independent variables as they tend to their respective limits. For instance take



and find the limit when 

If we put  and let 

 while putting  leads to a limit . Similarly, letting 

while *y* remains constant or vice-versa leads to zero limit. Thus, we are led to erroneous results. Geometrically speaking when we approach the point

(0, 0) along different paths, first along lines with slopes and and then along lines parallel to the coordinate axes, the function reaches different limits. The simultaneous limit postulates that by whatever path the point is approached, the function *f* attains the same limit. In general the determination whether a simultaneous limit exists or not is a difficult matter but very often a simple consideration enables us to show that the limit does not exist.

It may however be noted that

Non-existence of limit : The above remark makes it amply clear that if and if is any function such that when then *L*.

Thus, if we can find two functions and and and question does not exist.

Example 3 : Let

Solution: If we approach the origin along any axis, $f(x, y) = 0$. If approach (0,0) along any line $y = mx$, then

as

So any straight line approach gives,

But putting

which is different for the different *m* selected.

Hence,

Thus, the function possesses no limit at the origin, but a straight line approach gives the limit zero.



CHECK YOUR PROGRESS 2

Q 1: Show, by using the definition that

Q 2: Show that

(a) (b)

(c)

(d)

Continuity of function of several variables :

Definition : A function *f* is said to be continuous at a point s domain of definition, if

In other words, a function *f* is said to be continuous at a point of its domain of definition if for every *N* of

for all

(11)

Example 4 : Show that the function

is continuous at the origin.

Solution: Let and

Then

if

or, if

Thus when

Therefore,

Hence, f is continuous at (0, 0).

Note :

1. The definition of continuity of a function f at a point that besides , f is defined in a certain neighbourhood of and moreover the limit of f when the value of f
2. A function which is not continuous at a point is said to be discontinuous there at.

Remarks: A point to be particularly noticed is that if a function of more than one variable is continuous at a point, it is continuous at that point when considered as a function of a single variable. To be more specific if a function f of two variables x, y is continuous at , then function of x at and y at

The converse however is not true, i.e., a function may be a continuous function of one variable when the others remain constant and yet not be a continuous function of all the variables.

For instance, consider a function f where

The function is not continuous at (0, 0) for exist. But

and

so that *f* is continuous at (0, 0) when considered as a function of a single variable *x* or that of *y*.

A function is said to be continuous in a region if it is continuous at every point of the same.

As in limits, it can be easily proved that the sum, difference, product and quotient (provided the denominator does not vanish) of two continuous functions are also continuous.

Example 5 : Investigate the continuity at (0, 0) of



Solution: Since 
is not continuous at (0, 0).

Example 6 : Investigate for continuity at (1, 2) of



Solution: Here,



Hence, the function is not continuous at (1, 2).

The point (1, 2) is a point of discontinuity of the function.

However, if the function has the value 5 at (1, 2), it was then continuous at the point.



CHECK YOUR PROGRESS 3

Q 1: Show that the following functions are continuous at the origin:

(a) 

(b) 

Q 2: Show that the following functions are discontinuous at the origin:

(a)

(b)

(c)

(d)

(e)

(f)

10.6 DIRECTIONAL DERIVATIVES

Definition : The derivative of the function *f* at

the unit vector

(12)

provided the limit exists.

The directional derivative is also denoted by

"the derivative of *f* at P_0 in the direction of *u*".

Example 7 : Using the definition of directional derivative, find the derivative of

at P_0 (1, 2) in the direction of the unit vector

Solution:

Therefore, the rate of change of $_0$ (1, 2) in the direction of the unit vector is

10.7 PARTIAL DERIVATIVES

Definition : The ordinary derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables constant is called the partial derivative of the function with respect to the variable. Partial derivative of $f(x, y)$ with respect to x is generally denoted by

and

when these limits exist.

The partial derivatives at a particular point

or

and or

Also,

in case the limit exists.

Example 7 : If then find and (1, 2).

Solution: Now

at (1, 2),

at (1, 2).

Higher partial derivatives : Notice that and functions of two variables, so they can also be partially differentiated.

For a function of two variables R^2 R there are four possibility :

Higher order partial derivatives are defined similarly :

Usually,

For the functions we will been countering the mixed partial derivatives will generally be equal. In fact, this is true whenever and continuous.



CHECK YOUR PROGRESS 4

Q 1: Find all the second partial derivatives of

Gradient Vector

Definition: The gradient vector (gradiant) of $f(x, y)$ at a point P_0 the vector

obtained by evaluating the partial derivatives of f at P_0.

Theorem 4 : If $f(x, y)$ is differentiable in an open region containing P_0 , then

(13)

the dot product of the gradient f at P_0 and u.

Proof : We begin with the line

(14)

through P_0 s increasing in the direction of the unit vector rule for differentiable of f

(Applying Eq. (14))

(Proved)

Example 7 : Using the gradient formula find the directional derivative of

at the point (2, 0) in the direction $v = 3i - 4j$

Solution: The direction of v is the unit vector obtained by dividing v by its length:

The partial derivatives of f are everywhere continuous and at (2, 0) are given by

The gradient of f at (2, 0) is

The derivative of f at (2, 0) in the direction of u is therefore

Evaluating the dot product in the formula

where u and

following properties :

1. The function f increases most rapidly when u is the direction of P in its domain, f increases most rapidly in the direction of the gradient vector P. The derivative in this direction is

2. Similarly, f decreases most rapidly in the direction of derivative in this direction is

3. Any direction u orthogonal to a gradient change in f because then equals



CHECK YOUR PROGRESS 5

Q 1: Find the derivative of the following function at P_0 in the direction of v:

(a)

(b)

(c)

(d)

Q 2: Find the directions in which the functions increase and decrease most rapidly at P_0. Then find the derivatives of the following functions in these directions :

(a)

(b)

(c)

Q 3: If find and .

Q 4: If when and $f(0, 0) = 0$, show that



10.8 LET US SUM UP

- The set R^n is defined as the set containing n-tuples of real numbers. That is, R^n is the Cartesian product of R given as :

  $R^n = R \times R \times \ldots \times$ (n times).

- The limit of the function ^{n}R, with N, at the point ^{n} is the number R, denoted as

every there exists

- A function *f* is said to be continuous at a point definition, if

In other words, a function *f* is said to be continuous at a point of its domain of definition if for every neighbourhood *N* of for all

The derivative of the function *f* at P_0 vector

provided the limit exists.

- The ordinary derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables constant is called the partial derivative of the function with respect to the variable. Partial derivative of *f* (*x, y*) with respect to *x* is generally denoted by or or

to *y* are denoted or or



10.9 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1 :

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

Check your progress 2 :

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Check your progress 3 :

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Check your progress 4 :

Ans to Q No 1:



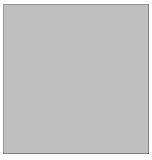
Check your progress 5 :

Ans to Q No 1: Try yourself.

Ans to Q No 2: Try yourself.

Ans to Q No 3: Try yourself.

Ans to Q No 4: Try yourself.



10.10 FURTHER READING

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10.11 MODEL QUESTIONS

Q 1: Show that the limit, when

(a) (b)

(c) (d)

Q 2: Show that the limit, when

(a) (b)

(c) (d)

Q 3: Discuss the following functions for continuity at (0, 0) :

(a)

(b) 

Q 4: Calculate 

(a) 

(b) 

Q 5: If 

find  and 

*** ***** ***

UNIT 11: THE RIEMANN INTEGRAL

UNIT STRUCTURE

11.1 Learning Objective

11.2 Introduction

11.3 Riemann Integrability

11.4 Riemann sums

11.5 Riemann's Condition for Integrability

11.6 Some classes of integrable Functions

11.7 Fundamental Theorem of Integral Calculus

11.8 Let Us Sum Up

11.9 Answers to Check Your Progress

11.10 Further Reading

11.11 Model Questions

11.1 LEARNING OBJECTIVES

In this unit, we will be able to learn about :

- partitions and Riemann sums
- upper and lower Riemann integrals
- riemann integrability
- riemann necessary and sufficient condition for R-integrability
- some classes of integrable functions
- fundamental theorem of integral calculus.

11.2 INTRODUCTION

The integration theory have arisen due to the necessity to evaluate the areas of plane regions. Likewise, integration of any function is interpreted to be the area under the graph of the function. The idea was to divide the plane region into various rectangles and find the limit of the sum of those rectangles when number of rectangles tend to infinity. This reference to integration from summation view point was always associated with

geometrical concepts. To formulate an independent theory of integration, the German mathematician, Riemann, gave a purely arithmetic treatment to the subject and thus developed the subject entirely free from the intuitive dependence on geometrical concepts. Afterwards, many developments and generalizations of the theory of integration followed. In this chapter, we define integration based on the definition by Riemann. However, this chapter is limited to finding integration over intervals and not any other sets. The function we discuss in this chapter is assumed to be bounded and unbounded functions are beyond the scope of this chapter. Accordingly, we begin by discussing the integration of real bounded functions on intervals and conclude the chapter by establishing the Fundamental theorem of Integral Calculus.

11.3 RIEMANN INTEGRABILITY

We begin by defining partition of an integral

Definition: Let

of [*a, b*] is a finite, ordered set *a, b*] such that

The points of *P* divide [*a, b*] into non-overlapping sub-intervals

We use the symbol to denote the sub-length of the sub-interval

As mentioned in the Introduction, in this chapter, we take function *f* to be a bounded real-valued function on [*a, b*]. Trivially, *f* is bounded on each sub-interval corresponding to each partition *P*. Let and supremum and infimum of *f* in

We define Upper and Lower sum of *f* corresponding to partition *P* as follows:

and

If M and m are the supremum and infimum of f in $[a, b]$, then

which implies

Putting $i = 1, 2, ..., n$ and adding all the inequalities, we get

(1)

For each partition P, there exist $L(P, f)$ and $U(P, f)$. We denote U to be the set of all upper sums of f for all the partitions of $[a, b]$ i.e.

is a partition of $[a, b]\}$

We denote L to be the set of all lower sums of f for all the partitions of $[a, b]$ i.e.

is a partition of $[a, b]\}$

Since both U(P, f) and L (P, f) are bounded as seen from (1), so the set U and L have supremum and infimum. The infimum of the set of upper sums is called the Upper integral and the supremum of the set of lower sums is called the Lower integral over $[a, b]$ i.e.

and

Definition: The function f on $[a, b]$ is said to be Riemann Integrable over $[a, b]$ if

and the common value c of the integral is called the Riemann integral of f over $[a, b]$ denoted as

Note: If f is integrable over $[a, b]$, then it can be expressed as

Remark: The upper and lower integrals are defined for every bounded function but they may not necessarily be equal for every bounded function.

Functions whose upper and lower integrals are not equal are said to be non-integrable functions.

Example: Show that a constant function $f(x) = k$ is integrable over $[a, b]$ i.e.

For any partition P of the interval $[a, b]$, we have

and

So

and

Thus

which indicates that the function $f(x) = k$ is integrable and

Definition: For any partition P, the length of the largest sub-interval is called the norm of the partition and is denoted by

Definition: A partition P^* is said to be refinement of P if P^* i.e. every point of P is a point of P^*.

Theorem : If P^* is a refinement of a partition P, then for a bounded function f,

(i) $L(P^*, f)$

(ii) $U(P^*, f)$

Proof: (i) As the function is bounded over the entire interval $[a, b]$, it is bounded in every sub-interval

Let us suppose P^* contains just one point more than P and let this extra point be where i.

Let and f in the intervals respectively.

Clearly, and

Thus,

$L(P^*, f) - L(P, f) =$

Thus $L(P^*, f) \geq L(P, f)$

If P^* contains p points more than P, same steps can be repeated p times to arrive at

$$L(P^*, f) \geq L(P, f)$$

Similarly we can prove that $U(P^*, f) \leq U(P, f)$

Corollary: For a refinement P^* of P containing more than p points more than P and for all

$$L(P, f) \leq L(P^*, f) \leq L(P, f) +$$

$$U(P, f) \geq U(P^*, f) \geq U(P, f) -$$

Proof: Proceeding as in the above theorem, when P^* contains one point more than P, we get

$$L(P^*, f) - L(P, f)$$

Since for all

Similarly,

$$\leq L(P^*, f) - L(P, f)$$

Now by introducing additional points one after another, and repeating the above reasoning p times, we get,

$$L(P^*, f) \geq L(P, f)$$

and since $L(P, f) \leq L(P^*, f)$

$L(P, f) \leq L(P^*, f)$

Similarly the other result may be proved.

Theorem: For oany two partitions P_1 and P_2,

i.e. no upper sum can be less than any lower sum.

Proof: Let P^* be a common refinement of P_1, P_2 so that

$P^* =$

Using the above theorem, we get

$L(P_1, f)$ $L(P^*, f)$ $U(P^*, f)$ $U(P_2, f)$

Thus $L(P^*, f)$ $L(P, f)$

Corollary: For any bounded function f,

Proof: Keeping P_2 fixed and taking supremum over all partitions P_1 of the above theorem, we get,

Similarly, taking infimum over all P_2 in the above equation, we get

Theorem: Darboux's Theorem: If f is a bounded function on $[a, b]$, then for every there corresponds

(i)

(ii)

for every partition P of $[a, b]$ with norm

Proof: (i) Since f is bounded, therefore, there exists $k > 0$ such that

Again, since the upper integral is the infimum of the set of upper sums, therefore for any there exists a partition $[a, b]$ such that

(2)

The partition P_1 has $p - 1$ points besides and .

Let be a positive number such that

Let P be any partition with norm and P^* be a refinement of P and P_1 such that $P^* = P$ $_1$.

Since P^* is a refinement of P having at most $p - 1$ more points than P, and for all s,

(P^*, f)

Also since we get, for any partition P with norm

(P^*, f)

(using equation (2))

since

The other part can be proved similarly.



CHECK YOUR PROGRESS 1

Q 1: Show that $U(P^*, f) - L(P^*, f)$ $U(P, f) - L(P, f)$ where P^* is a refinement of the partition P.

11.4 RIEMANN SUMS

In the previous section, we learned about integral of a function in terms of upper and lower sums. The numbers and these sums are not necessarily the values of the function f. However, in this section, we show that can also be considered as the limit of a sequence of sums in which and f.

Definition: Let P be a partition of $[a, b]$. Let us choose points such that . The sum

is said to be Riemann sum of f over $[a, b]$ relative to P.

Here are arbitrary and it can be any point whatsoever of

Definition: If for every there exist

for every partition $a, b]$ with norm every choise of points $S(P, f)$ is said to converge to A as i.e.



Definition: The second definition of integrability A function f is said to be integrable on [*a, b*] if  exists as 



Note: Since  when  therefore 

by 

The equivalence of the two definitions of integrability will now be established as follows :

Let us suppose, a bounded function *f* is integrable according to the first definition of integrability i.e.



We need to show that the function *f* is integrable according to the second definition of integrability i.e. for any  there exist  that for every partition *P* with 





Now, let 

By Darboux's theorem, there exists 



(3)

 (4)

for every partition *P* with 

If  is any point of 

 (5)

From (3), (4) and (5), we obtain that for any  there exist  such that for every partition *P* with 





Thus the function *f* is integrable according to the second definition of integrability.

Now, let us suppose the function *f* which is integrable according to the second definition of integrability i.e.

 exists

Thus for every  there exists 

 with norm  and for every choice of point  in 

there exist a number *A*, such that



It will be first shown that *f* is bounded.

Let if possible, *f* be not bounded.

Taking  *P* such that for every choice of

 in 





As *f* is not bounded in [*a,b*], it must also be so in at least one sub-interval say 

Let us take  when  so that every  except 

accordingly every term of the sum  except the term 

also fixed. Since *f* is not bounded in  we can choose a point  in 

such that



and thus we arrive at a contradiction.

Hence the function *f* is bounded on [*a,b*].

Now, let  be any positive number. So there exist 

for all partition *P* with 

(6)

We choose one such *P*. Taking the point and taking the supremum and infimum of the numbers $S(P, f)$ obtained as such, we get from (6),

Also since

Therefore

(7)

But .

Thus (7) implies,

so that the function is integrable.

Example : Compute where

Let *n* equal sub-intervals,, each of length

as

Let when

Thus when

Since the limit exists, so the function is integrable and



CHECK YOUR PROGRESS 2

Q 1: Show that where

Q 2: Show that where

11.5 RIEMANN'S CONDITION FOR INTEGRABILITY

In this section, we discuss the necessary and sufficient conditions for the integrability of a bounded function f is that to everey corresponds such that for every partition P of $[a, b]$ with norm

Proof: Necessary part: Let us suppose that the bounded function f is integrable i.e.

Let be any number. By Darboux's Theorem, there exists such that for every partition P with norm we have

(8)

(9)

Or

(10)

Adding equation (8) and (10), we get



for every partition *P* with norm 

Sufficient part : Let  *P* with norm 



For any partition, *P*, we know that





But 
greater than 0, so





Thus *f* is integrable.

Note: Alternatively we can state that : A necessary and sufficient condition for the integrability of a bounded function *f* is that



when the form  *P* tend to 0.

Theorem 2: A bounded function *f* is integrable on | *a, b*| if and only if for every  *P* of | *a, b*| such that



Proof: Necessary part : Suppose the function *f* is integrable i.e.



Let Since the upper and lower integrals are the infimum and supremum respectively of the upper and lower sums, therefore  partitions  and  such that



and



Let *P* be the common refinement of  and  i.e. 
Thus



Hence there exists a partition P such that 

Sufficient part: Let 

Let P be the partition for whch



For any partition P, we have



Thus



But 

than 0. Henceforth



which implies f is integrable.

11.6 SOME CLASSES OF INTEGRABLE FUNCTIONS

In this section we establish the Riemann integrability of some important classes of functions.

Theorem: Every continuous function on [a, b] is integrable on [a, b].

Proof: For  let us choose 



Since f is continuous on [a, b], therefore it is bounded and is uniformly continuous on [a, b] i.e. there exists 

 if  and 

(11)

We now choose a partition P with norm 

we have 







Henceforth *f* is integrable by Theorem 2.

Theorem: If a function *f* is monotonic on [*a, b*], then it is integrable on [*a, b*].

Proof : A monotonic function can either be increasing or decreasing. Let us suppose the function *f* is increasing.

For [redacted] let us choose [redacted] such that [redacted]

We choose a partition [redacted] *a, b*] with norm [redacted]

Since *f* is monotonic increasing, thus [redacted]

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

Hence *f* is integrable.

11.7 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

We now prove a theorem known as the first Fundamental theorem of Calculus.

Theorem: If function *f* is bounded and integrable on [*a, b*], then the function *F* defined as

[redacted]

is continuous on [*a, b*], and furthermore, if *f* is continuous at a point [redacted] then *F* is derivable at *c* and

[redacted]

Proof: Since function *f* is bounded, therefore there exists a number $M > 0$ such that

[redacted]

If and $a, b]$ such that

Thus for any

if

Hence the function F is continuous on $[a, b]$.

Now, given f is continuous at a point i.e. for any s such that

whenever

Let

Thus

Thus f at any point of $[a, b]$ implies derivability of F at that point.

Note: Since c is any point of $[a, b]$, we have for all

i.e. continuity of f on $[a, b]$ implies derivability of F on $[a, b]$

Definition: A derivable function F, if it exists such that its derivative is equal to a given function f is called a primitive of f.

Remark: Continuity of a function is not a necessary condition for the existence of a primitive.

Another theorem we state below is said to be the Second Fundamental theorem of Calculus.

Theorem: If a function f is bounded and integrable on $[a, b]$ and there exists a function F such that $a, b]$, then,

Proof: The function

there exists a

with norm

Or

(12)

for every choice of points in

We choose in

we have

Hence from (12), we get



11.8 LET US SUM UP

- Let *a, b*] is a finite, ordered set *a, b*] such that

  The points of *P* divide [*a, b*] into non-overlapping sub-intervals

  We use the symbol to denote the sub-length the interval

- Let and *f* in

  We define Upper and Lower sum of *f* corresponding to partition *P* as follows:

and



- The infimum of the set of upper sums is called the Upper integral and the supremum of the set of lower sums is called the Lower integral over [*a, b*] i.e.



and 

- The function *f* on [*a, b*] is said to be Riemann Integrable over [*a, b*] if



and the common value *c* of the integral is called the Riemann integral of *f* over [*a, b*] denoted as 

- For any partition *P*, the length of the largest sub-interval is called the norm of the partition and is denoted by 



- A partition P^* is said to be refinement of P if  P^* i.e. every point of *P* is a point of P^*.
- *f* P^* is a refinement of a partition *P*, then for a bounded function *f*,
  (i) $L(P^*, f)$ 
  (ii) $U(P^*, f)$ 
- For any two partitions  and 



i.e. no upper sum can be less than any lower sum.

- For partition *P* of [*a, b*] and points 

 the sum



is said to be Riemann sum of *f* over [*a, b*] relative to *P*.

- The second definition of integrability : A function *f* is said to be integrable on [*a, b*] if  exists as 

- A necessary and sufficient condition for the integrability of a bounded function f is that to every there corresponds every partition P of $[a, b]$ with norm
- A bounded function f is integrable on $[a, b]$ if and only if for every , there exists a partition P of $[a, b]$ such that
- Every continuous function on $[a, b]$ is integrable on $[a, b]$.
- If function f is bounded and integrable on $[a, b]$, then the function F defined as
  is continuous on $[a, b]$, and furthermore, if f is continuous at a point then F is derivable at c and
- /If a function f is bounded and integrable on $[a, b]$ and there exists a function F such that $a, b]$, then,



11.9 ANSWERS TO CHECK YOUR PROGRESS

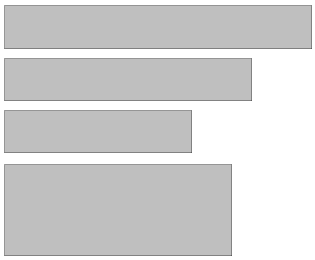
Check your Progress 1

Ans to Q No 1: Try Yourself.

Check your Progress 2

Ans to Q No 1: The function f is bounded and continuous on [– 1, 1] and

Let the partition
1, 1] into $2n$ equal sub-intervals each of length $1/n$, so that
as



Let  and  and let  and  for 







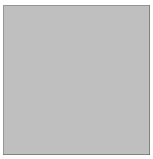


Thus 

and since theh limit exists, the function is integrable and



Ans to Q No 2: Try Yourself.



11.10 FURTHER READING

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11.11 MODEL QUESTIONS

Q 1: Prove that the function *f* defined as

 when 

and $f(0) = 0$ is integrable on [0, 1] although it has an infinite number of points of discontinuity.

Q 2: Show that the function $|x|$, where $|x|$ denote the greatest integer function is integrable on [0, 3] and



*** ***** ***

UNIT 12 : THE RIEMANN-STIELTJES INTEGRAL

UNIT STRUCTURE

12.1 Learning Objectives

12.2 Introduction

12.3 Generalization of Reimann Integral

12.4 Riemann-Stieltjes Integral as a Limit of Sum

12.5 Some Classes of Reimann-Stieltjes Integrable Functions

12.6 A relation Between R-integral and RS-integral

12.7 Let Us Sum Up

12.8 Answers to Check Your Progress

12.9 Further Reading

12.10 Model Questions

12.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- learn about a generalization of the Riemann integral
- know about lower and upper Riemann-Stieltjes integrals
- know about some classes of Riemann-Stieltjest integrable functions
- find the difference between Riemann integral and Riemann-Stieltjes integral.

12.2 INTRODUCTION

In this previous unit, we discussed about the theory of Riemann integration. In this unit, we generalize the theory and study the extensions due to Stieltjes, known as Riemann-Stieltjes integration. Throughout the unit, the function f we consider, will be real-valued bounded function on the domain of definition I : = [a, b]. Unlike the previous unit, here we introduce another monotonic increasing function

12.3 GENERALIZATION OF RIEMANN INTEGRAL

As stated in the introduction, let f be a bounded function on [a, b] and

We begin by defining

corresponding to any partition a, b].

Since is a monotonic increasing function, hence

Let M_i and m_i be the supremum and infimum of f in

We define Upper and Lower sum of f corresponding to partition P as follows :

and

If M and m are the supremum and infimum of f in [a, b], then

which implies

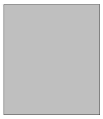
Putting i = 1, 2, ..., n and adding all the inequalities, we get

For each partition P, there exist L (P, f, set of all upper sums of f for all partitions of [a, b] and L to be the set of all lower sums of f for all the partitions of [a, b]. Since both U (P, f, , f, and infimum. The infimum of the set of upper sums is called the Upper integral of f with respect to is called the Lower integral of f with respect to

and 

These two integrals may or may not be equal. Definition : The function f is said to be Integrable with respect to 



and the common value of the integral is called the Riemann-Stieltjes integral of f with respect to , over [a, b] denoted as 

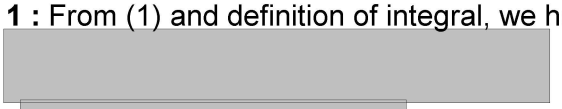
Note : The upper and lower integrals are defined for every bounded function but they may not necessarily be equal for every bounded function. Functions whose upper and lower integrals are not equal are said to be non-integrable functions.

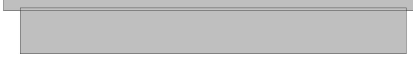
Note : If f is integrable with respect to

 or simply R(

Note : The Riemann-Stieltjes integral reduces to Riemann integral when (x) = x.

Remark 1 : From (1) and definition of integral, we have





Refinement of Partitions

Like in the previous chapter, we denote P* to be the refinement of partition P. The following theorem is stated as follows :

Theorem : If P* is a refinement of P, then

(i) L (P*, f , )  L (P, f, 

(ii) U (P*, f, )  U (P, f, 

Proof : (ii) Let p : = (a = x_0, x_1, ...x_{n-1}, x_n = b) be a partition of the given interval.

Let us suppose P* contains just one point more than P and let this extra point be  where 

Since f is bounded over the interval [a , b], so it is bounded on

every sub-interval

Let W1, W2, and Mi be the supremum of f in the intervals

respectively.

Clearly and

Thus,

=

Thus,

If P^* contains m points more than P, same steps can be repeated m times to arrive at

Theorem : A function f is integrable with respect to

if for every

Proof : Necessary Part : Let

Let

are the infimum and the supremum, respectively, of the upper and the lower sums, therefore there exists partitions P_1 and P_2 such that

Let $_1$ and P_2.

Sufficient part : Let

For any partition P, we know that

But

possible but greater than 0, so

Thus Let



CHECK YOUR PROGRESS 1

Q 1: If , then prove that

Q 2: For any two partitions P_1 and P_2.

12.4 RIEMANN-STIELTJES INTEGRAL AS A LIMIT OF SUM

As an analog to the Riemann sum, in this section, we introduce the sum S (P, f,

of a Riemann-Stieltjes integral.

Definition : We define the sum S (P, f,

corresponding to a partition P of [a, b] and

Definition : We say that S (P, f,) coverges to A as

if for every , there exists

of [a, b] with $_i$ in $[x_{i-1}, x_i]$.

Theorem : If exists, then

and

Proof : Let us suppose

= A

Thus for , there exists

[a, b] with $_i$ and $[x_{i-1}, x_i]$, we have

or

Let P be one such partition. If we let the points t_i range over the intervals $[x_{i-1}, x_i]$ and take the infimum and supremum of the sum S (P, f,) then (6) gives,

Again, since S (P, f,) and lie between and ,

so we have

12.5 SOME CLASSES OF RIEMANN-STIELTJES INTEGRABLE FUNCTIONS

Theorem : If f is continuous on [a, b], then

for every there corresponds

for every partition $P = \{a = x_0, ..., x_n = b\}$of [a, b] with

every choice of

Proof : If is assumed that all functions are bounded and is monotonically increasing. Let be given. Let us choose

Since continuity of f on [a, b] implies uniform continuity on [a, b], therefore for , there exists

if

Let P be a partition of [a, b] with

Therefore,

Thus

Again if , then for , there exists all partitions P with and for all positions with therefore we have

Thus we have

Note : Continuity is a sufficient condition but not necessary condition for integrability of a function.



CHECK YOUR PROGRESS 2

Q 1: If f is monotonic on [a, b] and is monotonic as well as continuous on [a, b] then .

12.6 A RELATION BETWEEN R-INTEGRAL AND RS-INTEGRAL

The following theorem provides a relation between the Riemann integral and Riemann-Stieltjes integral

Theorem : If and is monotonically increasing on [a, b] such that then

Proof : Let .

Since f is bounded, there exists M > 0 such that

Again, since , therefore

consequently there exists ,

for and all

Now, for and all

Let and P be any partition with

Then for all

are points

Thus,

=

Hence for any , there exists

with , (9) holds which implies

equals

i.e.

Theorem : If f is continuous on [a, b] and
[a, b], then

Proof : Under the given conditions, all the integrals exist.

Let $P = \{a = x_0, ..., x_n = b\}$ be any partition of [a, b]. Thus by Lagrange's Mean Value theorem it is possible to find that

i.e.

Thus

Proceeding to limits as



12.7 LET US SUM UP

- We define Upper and Lower sum of f corresponding to partition P as follows :

and

- If M and m are the supremum and infimum of f in [a, b], then

which implies

Putting i = 1, 2, ..., n and adding all the inequalities, we get



- For each partition P, there exist  and . We denote Uto be the set of all upper sums of f for all the partitions of [a, b] and L to be the set of all lower sums of f for all the partitions of [a, b]
- The infimum of the set of upper sums is called the Upper integral of f with respect to

  called the Lower integral of f with respect to



and 

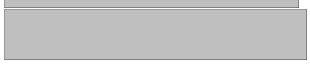
- The function f is said to be Integrable with respect to  sense if



and the common value of the integral is called the Riemann-Stieltjes integral of f with respect to , over [a, b] denoted as .

- The upper and lower integrals are defined for every bounded function but they may not necessarily be equal for every bounded function. Functions whose upper and lower integrals are not equal are said to be non-integrable functions.
- The Riemann-Stieltjes integral reduces to Riemann integral when (x) = x.
- If P^* is a refinement of P, then

  (i) 

  (ii) 

- A function f is integrable with respect to 

  every 



- The sum 



corresponding to a partition P of [a, b] and

- coverges to A as

  if for every , there exists

  [a, b] with and every choice of in

- If

  and

- If f is continuous on [a, b] then

  , there corresponds

  for every partition $P = \{a = x_0, ..., x_n = b\}$ of [a, b] with

  every choice of

- If [a, b] and

  , then

- If f is continuous on [a, b] and

  then



12.8 ANSWERS TO CHECK YOUR PROGRESS

Check your Progress 1

Ans to Q No 1: Try Yourself.

Ans to Q No 2: Try Yourself.

Check your progress 2 :

Ans to Q No 1: Let .

For any integer n > 0, we choose a partition P = {a = x_0, ..., x_n = b}of [a, b] such that

This is possible since

continuous on [a, b]. Thus assumes every value between its bounds (a) and

Let f be monotonically increasing on [a, b], so that its lower and upper bound m_i, M_i, in [x_{i-1}, x_i] are given by

Thus

<

i.e.

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12.10 MODEL QUESTIONS

Q 1: A function
where
and for
Prove that over [a, b] and

*** ***** ***

UNIT 13 : UNIFORM CONVERGENCE OF SEQUENCES AND SERIES

UNIT STRUCTURE

13.1 Learning Objectives

13.2 Introduction

13.3 Pointwise Convergence

13.4 Uniform Convergence

13.5 Cauchy's General Principle of Uniform Convergence

13.6 Test for Uniform Convergence

13.7 Uniform Convergence and Continuity

13.8 Uniform Convergence and Integration

13.9 Uniform Convergence and Differentiation

13.10 Let Us Sum Up

13.11 Answers to Check Your Progress

13.12 Further Reading

13.13 Model Questions

13.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- learn about uniform convergence of sequence and series
- know about the Cauchy's general principle of uniform convergence
- determine the tests for uniform convergence
- learn about relation between Uniform convergence and continuity
- learn about relation between Uniform convergence and integration
- learn about relation between Uniform convergence and differentiation.

13.2 INTRODUCTION

In the previous units, we learned about the sequences and series. Usually, we considered those sequences and series whose terms are numbers. In this chapter, we shall study about those sequences and series

whose terms depend ona variable i.e. the terms are real-valued functions defined on an interval as domain. The terms of such sequences and series are denoted by $f_n(x)$ and the sequences and series are of the form $\{f_n\}$ and respectively.

13.3 POINTWISE CONVERGENCE

We start the section by defining point-wise limit of the sequence $\{f_n\}$ and series

of problems which arise when sequences (viz. series) of variable terms and the limit function f of the terms f_n are considered. We investigate under what supplementary conditions, the properties of the terms f_n are transferred to the limit function f. Henceforth, in this respect, in the next section, we introduce the concept of uniform convergence of sequence (viz. series) in the domain of definition.

Let $\{f_n\}$ be a sequence of functions, defined on an interval I : = [a, b]. For each point with terms

We suppose that the sequence of numbers every converges to

Similarly, let the sequences at points to

Definition : A real-valued function f with domain I : = [a, b] and range set defined by (1) is said to be the limit or point-wise limit of the sequence $\{f_n\}$ on [a, b] and the sequence $\{f_n\}$ is said to be pointwise convergent to f on [a, b] if

Similarly, if the series converges for every point

as the sum or the pointwise sum of the series

Definition : f is said to be the point-wise limit of the sequence of functions $\{f_n\}$ defined on [a, b], if for each , and for each s an integer m such that

Now, a natural question arises that for a sequence (viz, series) of variable terms, whether properties belonging to terms viz. boundedness, continuity, integrability, differentiablility etc. are transferred to the limit function of the corresponding sequence (viz, series). The example below will give the answers.

Example 1 : The geometric series

$1 + x + x^2 + ...$

converges to $(1 – x)^{-1}$ in the interval $–1 < x < 1$.

All the terms

is not bounded.

Example 2 : We consider the series

and x is real At x = 0, each

For , is a geometric series with common ration

thus $f(x) = 1+ x^2$

Hence,

Thus each term of the series is continuous but the sum f is not continuous.

Example 3 : The sequence $\{f_n\}$ where

limit

Therefore, $f'(x) = 0$ and so $f'(0) = 0$

But

and so as

Thus at x = 0, the sequences

i.e. the limit of differentials is not equal to the differential of the limit.

Example 4 : The sequences where has the limit

At x = 0, each , so that

Thus the limit function for

Therefore

Again

so that

Thus

Thus the limit of integral is not equal to the integral of the limit. In other words, the sequence of integrals maynot converge to the integral of the limit of the sequence.

At the end of this section, we can conclude that a new categoryof problem arises with the consideration of sequences (viz. series) of variable terms. Thus to determine the conditions so that the properties of the terms f_n are transferred to the limit function f, we introduce the concept of Uniform convergence of sequences (viz. series) in the next chapter.

13.4 UNIFORM CONVERGENCE

Definition : A sequence of functions $\{f_n\}$ is said to be converge uniformly to a function f on an interval [a, b], if for any , and for all exists an integer N (independent of x but dependent on

,

From the definition, it is clear that every uniformly convergent sequence is pointwise convergent and the uniform limit function is same as the pointwise limit function. However the converse is not always true.

Definition : A series of function

b] if the sequence $\{S_n\}$ of its partial sums, defined as

converges uniformly on [a, b]

13.5 CAUCHY'S GENERAL PRINCIPLE OF UNIFORM CONVERGENCE

Theorem : A sequence of functions $\{f_n\}$ defined on [a, b] converges uniformly on [a, b] if and only if for every and for all integer N such that

Proof : Necessary Part : Let the sequence $\{f_n\}$ uniformly converges on [a, b] to the limit function F i.e. for given and for all s an integers m_1 such that

Similarly for the same and for all integers m_2 such that

Let N = max $\{m_1, m_2\}$

Therefore

Sufficient part : Let for and for all integers N such that

By Cauchy's general principle of converge (stated in chapter 3), $\{f_n\}$ converges for each

converges pointwise t f.

We now prove the convergence is uniform.

For a given

We fix n and let in (2). Since as

which proves that

Remark : In the statement of the theorem, (2) may be equivalently replaced by

Example : Test the uniform convergence of the sequence $\{f_n\}$ where

R

The sequence converges pointwise to f where

R

Let $\{f_n\}$ converges uniformly in any interval [a, b] so that the point-wise limit is also the uniform limit. Therefore for given s an N such that

Taking and inteer m > N such that 1/m

and x = 1/m that

Thus, we arrive at a contradiction and so the sequence is not uniformly convergent in the interval [a, b] containing the point 1/m. But since

uniformly convergent on any interval [a, b] containing 0.



CHECK YOUR PROGRESS 1

Q 1: Show that the sequence $\{f_n\}$ such that

is uniformly convergent in any interval [0, b].

Q 2: Show that the series whose sun to n terms $S_n(x) = nxe^{-nx^2}$ is pointwise and not uniformly convergent on any interval [0, k].

13.6 TEST FOR UNIFORM CONVERGENCE

In the previous section, we got introduced with the concept of uniform convergence. We have used merely the definition of uniform convergence for the purpose of determining whether the sequence (viz. series) is uniformly converging in a given interval. In this section, we will inquire how we can determine whether a given sequence or a series does or doest ot converge uniformly in a given interval.

Theorem 1 : Let $\{f_n\}$ be a sequence of functions such that

and let Then

only if as

Proof : Necessary part : Let

, there exists an integer N such that

as

Sufficient part : Let as so that for any

there exists an integer N such that

Theorem 2 : Weierstrass's M-test A series of functions will converge uniformly on [a, b] if there exists a convergent series numbers such that for all

Proof : Let .

Since is convergent, therefore there exists a positive integer N such that

Hence, for all and for all

(3)

Equatiosn (3) and (4) imply that

Remark : The converse of the theorem 2 is not true in general.

Besides the Weierstrass's M-test, we have two other tests viz. Abel's and Dirichlet's test to test the uniform convergence of a series.

Theorem 3 : Abel's test If $b_n(x)$ is a positive, monotonic decreasing function of n for each fixed value of $_n(x)$ is bounded for all values of n and x concerned, and if the series [a, b], then so also is the series

Theorem 4 : Dirichlet's test If $b_n(x)$ is a monotonic function of n for each fixed value of $_n(x)$ tends uniformly to zero for if there is a number K > 0 independent of x and n, such that for all values of ,

then the series

Theorem : (A) If a sequence $\{f_n\}$ converges uniformly in [a, b] and x_0 is a point of [a, b] such that

then $\{a_n\}$ converges and

(B) If a series $_0$ is a point in [a, b] such that

then converges and

We now look at few examples to test the uniform convergence of sequence and series.

Example : Show that the sequence $\{f_n\}$ such that

is not uniformly convergent on any interval containing 0.

We have

Now attains maximum value 1/2 at x = 1/n. And . Taking an interval [a, b] containing 0, we have

which does not tend to 0 as

Hence the sequence $\{f_n\}$ is not uniformly convergent in any interval containing 0.



CHECK YOUR PROGRESS 2

Q 1: Prove that the sequence $\{f_n\}$ where

converges uniformly on any closed interval I.

13.7 UNIFORM CONVERGENCE AND CONTINUITY

In this section, we present theorems connecting continuity and uniform convergence of series (viz. sequences).

Theorem : (A) If $\{f_n\}$ is a sequence of continuous functions on an interval [a, b] and if

(B) If a series s terms f_n are continuous at a point x_0 of the interval, then the sum function f is also continuous at x_0.

Proof : (B) Since

for , we can choose N such that for all

and in particular, at a point

Again, since each f_n is continuous at x_0, the sum of a finite number of functions $_0$. Therefore, for there exists

for

Hence for

which implies when is continuous at

Corollary : If a series

and if the functions f_n are all continuous throughout the interval, then so is the sum function f.

Proof : The proof follows from the fact that if x_0 is an arbitrary point of [a, b], then the above theorem holds for all points of [a, b].

Remark : The converse of the theorem may not be true i.e. series (viz. sequence) of continuous terms exist which have a continuous sum (limit) but are not uniformly convergent. In other words, the condition of uniform convergence is only sufficient, but not necessary.

Theorem : Dini's theorem on uniform convergence (A) If a sequence of continuous functions $\{f_n\}$, defined on [a, b] is monotonic increasing, and converges (pointwise) to a continuous function f, then the convergence is uniform on [a, b].

(B) If the sum function of a series continuous terms defined on an interval [a, b] is continuous on [a, b], then the series is uniformly convergent on the interval.

Example : Show that the series

is not uniformly convergent on [0, 1].

The terms of the series are continuous and the series converges pointwise to f where

Hence the series does not converge uniformly on [0, 1].



CHECK YOUR PROGRESS 3

Q 1: Show that the series is uniformly convergent on any interval [a, b] but not pointwise convergent on [0, b] where 0 < a < b.

13.8 UNIFORM CONVERGENCE AND INTEGRATION

In this section, we present theorem connecting the integrability and uniform convergence of sequence (viz.) series.

Theorem : (A) If a sequence $\{f_n\}$ converges uniformly to f on [a, b], and each function f_n is integrable, then f is integrable on [a, b], and the sequence converges uniformly to

(B) If a series term $f_n(x)$ is integrable, then f is integrable on [a, b], and the series converges uniformly to

We then say that the series is integrable term by term.

Proof : (A) Let .

By the uniform convergence of the sequence, there exists an integer m such that for all

In particular,

For this fixed m, since f_m is integrable, we choose a partition P of [a, b], such that

From (5), we have

Again from (5), we have

From (6), (7) and (8), we get

Thus f is integrable on [a, b]

We now prove that the sequence

to

Since the sequence $\{f_n\}$ converges uniformly to f, therefore for ,
there exists an integer N such that for all

Then for all and

This implies converges uniformly to
b] i.e.

13.9 UNIFORM CONVERGENCE AND DIFFERENTIATION

In this section, we present theorem connecting the differentiability and uniform convergence of sequence (viz.) series.

Theorem : Let $\{f_n\}$ be a sequence of differentiable functions on [a, b] such that it converges at least at one point
differentials converges uniformly to G on [a, b], then the given sequence $\{f_n\}$ converges uniformly on [a, b] to f and $f'(x) = G(x)$.

(B) Let
that it converges at least at one point . If the series of differentials
converges uniformly on [a, b] to f and $f'(x) = G(x)$.

Proof : Let .

By the convergence of

for , we can choose a positive integer N such that for all ,

(9)

(10)

Applying Lagrange's mean value theorem to the function ... for any two points x and t of [a, b], we get for , for all .

(11)

(13)

and

Thus, the sequences $\{f_n\}$ uniformly converges on [a, b]

Let it converge to f.

Now, for a fixed x on [a, b] and for

and

Since each f_n is differentiable, therefore for each n,

Thus

so that converges uniformly on [a, b] for

Since $\{f_n\}$ also converges uniformly on f, therefore

Thus converges uniformly on [a, b] for

Also, form (14), we get

Thus exists and therefore f is differentiable and .

Hence

We state the simple version of the theorem for series, that for sequence too is on the same line.

Theorem : Let a series of differentiable functions converges pointwise to f on [a, b] and each is continuous on [a, b], and the series converges uniformly to G on [a, b], then the given series converges uniformly to f on [a, b], and



13.10 LET US SUM UP

- A real-valued function f with domain I : [a, b] and range set defined by 91) is said to be the limit or point-wise limit of the sequence $\{f_n\}$ on [a, b] and the sequence $\{f_n\}$ is said to be pointwise convergent to f on [a, b] if
- Similarly, if the series converges for every point , we define

  as the sum or the pointwise sum of the series
- f is said to be the point-wise limit of the sequence of functions $\{f_n\}$ defined on [a, b], if for each , and for each s an integer m such that
- A sequence of functions $\{f_n\}$ is said to be converge uniformly to a

function f on an interval [a, b], if for any , and fr all exists an integer N (independent of x but dependent on for all

- Every uniformly convergent sequence is pointwise convergent and the uniform limit function is same as the pointwise limit function. However the converse is not always true.
- A series of function sequence $\{S_n\}$ of its partial sum, defined as

  converges uniformly on [a, b].
- A sequence of functions $\{f_n\}$ defined on [a, b] converges uniformly on [a, b] if and only if for every and for all integer N such that
- Let $\{f_n\}$ be a sequence of functions such that

  and let Then if and only if as
- Weierstrass's M-test : A series of functions uniformly on [a, b] if there exists a convergent series numbers such that for all
- Abel's test : If $b_n(x)$ is a positive, monotonic decreasing function of n for each fixed value of $_n(x)$ is bounded for all values of n and x concerned, and if the series on [a, b], then so also is the series
- If a sequence $\{f_n\}$ converges uniformly in [a, b] and x_0 is a point of [a, b] such that

then $\{a_n\}$ converges and

- If a series converges uniformly to f in [a, b] and x_0 is a point in [a, b] such that

  then converges and

- Dirichlet's test : If $b_n(x)$ is a monotonic function of n for each fixed value of $_n(x)$ tends uniformly to zero for if there is a number K > 0 independent of x and n, such that for all values of

  then the series

- If $\{f_n\}$ is a sequence of continuous functions on an interval [a, b], and if

- If a series $_n$ are continuous at a point x_0 of the interval, then the sum function f is also continuous at x_0.

- Dini's theorem on uniform convergence : If a sequence of continuous functions $\{f_n\}$ defined on [a, b] is monotonic increasing, and converges (pointwise) to a continuous function f, then the convergence is uniform on [a, b].

- If the sum function of a series terms defined on an interval [a, b] is continuous on [a, b], then the series is uniformly convergent on the interval.

- If a sequence $\{f_n\}$ converges uniformly to f on [a, b] and each function f_n is integrable, then f is integrable on [a, b], and the sequence converges uniformly to

- If a series $_n(x)$

is integrable, then f is integrable on [a, b], and the series

converges uniformly to

- Let $\{f_n\}$ be a sequence of differentiable functions on [a, b] such that it converges at least at one point differentials sequence $\{f_n\}$ converges uniformly on [a, b] fo f and $'x = G(x)$.
- Let converges at least at one point . If the series of differentials converges uniformly to G on [a, b], then the given series converges uniformly on [a, b] to f and $f'(x) = G(x)$.
- Let a series on [a, b] and each is continuous on [a, b], and the series converges uniformly to G on [a, b], then the given series converges uniformly to f on [a, b], and $f'(x) = G(x)$.

13.11 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1 :

Ans to Q No 1: We have

and hence the sequence converges pointwise to 0.

Now, for any ,

if , which decreases with x, the maximum value being .

Let N be an integer greater than or equal to , so that for there exists N such



Hence the sequence converges uniformly in [0, b].

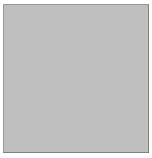
Ans to Q No 2: Try Yourself.

Check your Progress 2

Ans to Q No 1: Try Yourself.

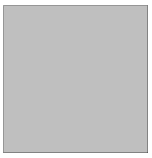
Check your Progress 3

Ans to Q No 1: Try Yourself.



13.12 FURTHER READING

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13.13 MODEL QUESTIONS

Q 1: Show that the series 

real x.

Q 2: Show that the series 

.

Q 3: Show that is uniformly convergent in [0, 1] if converges.

Q 4: Prove that the series 

*** ***** ***

UNIT 14 : CONVERGENCE OF IMPROPER INTEGRALS

UNIT STRUCTURE

14.1 Learning Objectives

14.2 Introduction

14.3 Convergence of Improper Integrals of the First Kind

14.4 Tests for Convergence of Improper Integrals of the First Kind

14.5 Absolute Convergence

14.6 Convergence of Improper Integrals of the Second Kind

14.7 Tests for Convergence of Improper Integrals of the Second Kind

14.8 Let Us Sum Up

14.9 Answers to Check Your Progress

14.10 Further Reading

14.11 Model Questions

14.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- know about the convergence of improper integrals
- learn about the various test for convergence of improper integrals of the first kind
- know about absolute convergence
- learn about the tests for convergence of improper integrals of the second kind.

14.2 INTRODUCTION

In the previous unit on Riemann Integral, the theory of integration was developed under two assumptions.

(i) the interval of integration was required to be a closed and bounded interval, and

(ii) the integrand was required to be bounded on the interval.

If these restrictions are relaxed on the theory of integration, we

have the followed two types of integrals, called improper integrals or infinite integrals.

(i) improper integrals on a finite interval where the integrand is unbounded.

(ii) improper integrals on an unbounded interval.

Thus



are few examples of improper integrals where the restriction on the interval of integration is removed and the integrand is not bounded.

Based on the relaxation on the above conditions, we can classify the improper integral as follows :

Improper integral of the first kind : An improper integral with finite interval of integration [a, b] with its integrand having a finite number of points of infinite discontinuity

Improper integral of the second kind : An improper integral with infinite interval of integration and bounded integrand.

Improper integral of the mixed kind : An improper integral with infinite interval of integration with its integrand having a finite number of points of infinite discontinuity.

We define convergence of improper integrals of the first and second kind and discuss the properties of each type separately in the succeeding sections.

14.3 CONVERGENCE OF IMPROPER INTEGRALS OF THE FIRST KIND

Case I. Convergence of the improper integral 
only point of infinite discontinuity of f in [a, b].

Let te left end point a of the closed and bounded interval [a, b] be the only point of infinite discontinuity of a function f whcih is bounded and integrable on  for every  satisfying 

Let



If exists (finitely) then the improper integral is said to be convergent and we write where l is the limit

If does not exist (finitely) then the improper integral is a said to be divergent.

Note : If a be the only point of infinite discontinuity of a function f which is bounded and integrable on for every satisfying and is convergent, then

Example : The integral discontinuity of the integrand.

Solution : The integrand is bounded and integrable on for all satisfying ,

Therefore the improper integral

Case II. Convergence of the improper integral when b is the only point of infinite discontinuity of f in [a, b].

Let the right end point b of the closed and bounded interval [a, b] be the only point of infinite discontinuity of a function f which is bounded and integrable on for every satisfying

Let

If exists (finitely then the improper integral is a said to be convergent. If the limit be *l*, we write

If does not exist (finitely) then the improper integral is a said to be divergent.

Note : If b be the only point of infinite discontinuity of a function f which is bounded and integrable on for every satisfying

and  is convergent, then 

.

Example : The integral 

discontinuity of the integrand.

Solution : The integrand is bounded and integrable on  for all 

satisfying  ,

Therefore the improper integral 

Example : The integral 

discontinuity of the integrand.

Solution : The integrand is bounded and integrable on  for all 

satisfying  ,

Therefore the improper integral 

.

Case III. Convergence of the improper integral 

the only point of infinite discontinuity of f in [a, b]

Let the end points a, b of the closed and bounded interval [a, b] be the only points of infinite disontinuity of a function f which is bounded and ingegrable on  for every 

.

Let 

If the improper ingegrals  and 

convergent according to the definitions given above, then the improper integral 



Note : If the end points a and be of the closed and bounded interval [a, b] be the only points of infinite discontinuity of a function f and the improper integral  be convergent, then for any point 



Example : The integral  is improper, since 0 and 2 are points of infinite discontinuity of the integrand.

Solution : The integrand is bounded and integrable on 

 satisfying 

Let us examine if 

 exist.





Therefore  is convergent and 

Case IV. Convergence of the improper integral 

number of points c_1, c_2, ...c_n, and finite number of points of discontinuity of f in [a, b], where 

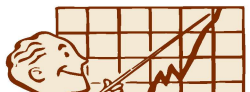
If the improper integrals



all exist in accordance with the definitions given above and we regard

then we say that



Let

Then

since f(x) > 0 on

. This shows that

decreasing function on (0, b – a). Therefore will tend to a finite limit as

if and only if

In other words, the improper integral

and only if there exists a positive real number k such that

for all f satisfying

Note : If be not bounded above, then tends to as

the improper integral diverges to

Theorem : Comparison test

Let a be the only point of infinite discontinuity of the functions f and g which are both integrable on for all satisfying

and for all

(i) is convergent is convergent

(ii) is divergent is divergent

Proof : Since f and g both are integrable on and

for all , we have

This holds for all satisfying

(i) If

real number k/ such that

for all satisfying

Then for satisfying

proves that

(ii) Let Then

If be divergent, then

(0, b – a) and therefore is not bounded above on (0, b – a).

This proves that

This completes the proof.

A useful comparison integral :

The integral is convergent if and only if

Proof : The integral is proper if

Let . Let

infinite discontinuity of f. f is integrable on for

(x) > 0 for all

Let ,

If

If , and if

If ,

Since exists finitely when

, when , the improper integral

convergent if an only if

Since the integral is proper if

integral is convergent if an only if

integral  is convergent if and only if 

Theorem : 

Let a be the only point of infinite discontinuity of a function f which is integrable on  for  and f (x) > 0 for all 

If  *l* is a non-zero finite number, then the integral  is convergent if and only if 

Proof : Since  for all  *l* > 0

Let us choose a positive  such that 

Since  that

 for all 

or,  for all 

If , the integral 

If , the integral  is divergent. Since  if followsfrom the left hand inequality that  is divergent.

Therefore,  is convergent if and only if 

thereore  is convergent if and only if 

Note : If 

the integral  is convergent if and only if 

Example : Show that 

are both positive.

Solution : Let the given integral be 

 and 

1 is the only point of infinite discontinuity of f if n < 1.

Let us examine the convergence

and the convergence of

Convergence of

f (x) > 0 for all .

zero finite number.

By test

if and only if m > 0.

Convergence of

f (x) > 0 for all .

zero finite number).

By test

and only if n > 0.

Therefore both the integrals

are convergent if and only if m > 0 and n > 0.

Hence

> 0.

Note : The integral a function and it is denoted by B (m, n)

Theorem : Cauchy

Let a be the only point of infinite discontinuity of a function f which isintegrable on for all satisfying

keep the same sign on

A necessary and sufficient condition for the convergence of the improper integral  is that for a pre-assigned positive exists a positive 

 for all  satisfying 

Proof : Let 

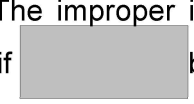
The improper integral  is convergent if  exists finitely. By Cauchy's creiterion for the exixtence of finite limits  exists finitely if and only if for a pre-assigned positive  corresponds a positive 

 for all  satisfying 

 for all  satisfying 

This conpletes the proof.

14.5 ABSOLUTE CONVERGENCE

Definition : The improper integral  absolutely convergent if  be convergent.

Theorem : An absolutely convergent improper integral  (where a is the only point of infinite discontinuity of f in [a, b] and f is integrable on  for all  satisfying 

Proof : Here the integral  only point of infinite discontinuity of f in [a, b]

Let . Then there exists a positive 

 for all  satisfying 

We also have 

Therefore  for all 

This proves that the integral

This completes the proof.

Notes : (1) The converse of the theorem is not true. We shall establish this by some examples.

(2) Since

applied to establish the convergence of the improper integral

Example : Show that the improper integral

Solution : Let

0 is the only point of infinite discontinuity of . f (x) does not keep the same sign in the interval

for all and

Therefore is convergent, i.e.

convergent and therefore



CHECK YOU PROGRESS 2

Q 1: Test the convergence of the following integrals :

a) b)

c)

Q 2: Show that if  is divergent if 

Q 3: Show that 

14.6 CONVERGENCE OF IMPROPER INTEGRALS OF THE SECOND KIND

Case I. Convergence of the improper integral 

integrable on [a, X] for all X > a.

Let 

If 

 is said to be convergent. If the limit be *I* , we write  *1*

If 

 is said to be divergent.

Example : Let us consider the integral 

integrable on any closed interval [0, X], X > 0. The integral is improper.

Let  Then 



Therefore the integral  is convergent and .

Example : Let us consider the integral 

integrable on any closed interval [0, X], X > 0. The integral is improper.

Let . Then 

. Therefore the integral  is divergent.

Case II. Convergence of the improper integral 

integrable on [X, b] for all X < b.

Let

If

is said to be convergent. If the limit be I, we write = I.

Case III. Convergence of the improper integral

integrable on $[X_1, X_2]$ for all $X_1, X_2 \in R$ satisfying $X_1 < X_2$

Let $c \in R$. If both the integrals and convergent according to the definitions I and II above, then the improper integral

Example : Let us consider the integral

integrable on any closed interval $[X_1, X_2]$, $X_2 > X_1$. The integral is improper.

Let us consider the integrals and where a

Let . Then

.

Therefore the improper integral

Let . Then

Therefore the improper integral

Consequently,

IV. Convergence of the improper integral 

number of points of infinite discontinuity $c_1, c_2, \ldots, c_m$.

Let $c_1 < c_2 < \ldots < c_{m.}$ If each of the integrals 

and 

then the improper integral 

write.

14.7 TESTS FOR CONVERGENCE OF IMPROPER INTEGRALS OF THE SECOND KIND

Theorem : Let a function f be integrable on [a, X] for all X > a and f (x) > 0 for all 

A necessary and sufficient condition for the convergence of the

improper integral 

such that

for all X > a

Proof : Let 

Let 

Then 

f (x) > 0 on $[X_1, X_2]$.

. This shows that 

increasing function on (a, ). Therefore 

if and only if 

In other words, the improper integral 

and only if there exists a positive real number k such that

for all X > a

Note : If be not bounded above, then (X) tends to as and the improper integral diverges to

Theorem : Comparison test

Let the functions f and g be both integrable on [a, X] for X > a and for all

(i) is convergent is convergent.

(ii) is divergent is divergent.

Proof : Since f and g both are integrable on [a, X] and all , we have

(i) If number k' such that

Then for all X > a and this proves that is convergent.

(ii) Let be divergent. Then

Since .

This proves that

This completes the proof.

A useful comparison integral :

The integral , where a > 0, is convergent if and only if .

Proof : Let

If , we have

and if ,

If ,

If

Since exists finitely when and when the improper integral is convergent, if and divergent if

This completes the proof.

Theorem :

Let f (x) > 0 for all . If *l*, where *l* is a non-zero finite number, the improper integral is convergent if and only if .

Proof : Let *l*. Then *l* > 0 .

Let us choose a positive *l* – *l*, there exists a positive real number k > a such that

for all

for all

for all

is convergent if *l* +

last ineqality that is convergent if

is divergent if *l* –

first ineqality that is divergent if

Therefore is convergent if and only if

Theorem : Cauchy

Let a

A necessary and sufficient condition for the convergence of the improper integral is that for a pre-assigned positive there exists a positive number X_0 such that

for all $X_1, X_2 > X_0$.

Proof : Let

This improper integral is convergent if exists finitely.

By Cauchy's criterion for the existence of finite limits exists finitely if and only if for a pre assigned positive there corresponds a positive X_0 such that

for all X_1, X_2 satisfying $X_1, X_2 > X_0$

for all X_1, X_2 satisfying $X_1, X_2 > X_0$

for all X_1, X_2 satisfying $X_1, X_2 > X_0$

This completes the proof.

Absolutely convergent :

Definition : The improper integral is said to be absolutely convergent if the integral

The improper integral

convergent if the integral

Theorem : An absolutely convergent improper integral (where f is bounded and integrable on [a, X] for every X > a) is convergent.

Proof : Here f is integrable on [a, X] for every X > a and the improper integral is absolutely convergent.

Then the integral

assigned positive there exists a positive

for all X_1, X_2 satisfying $X_1, X_2 > X_0$

We also have

Therefore for a pre-assigned positive

such that <

$_1, X_2$ satisfying $X_1, X_2 > X_0$.

This implies that the integral

This completes the proof.

Notes : (i) The converse of the theorem is not true. We shall establish this by some examples.

(ii) Since | f (x) | is always positive, comparison tests can be applied to establish the convergence of the improper integral .

Example : Examine the convergence of the improper integral , where

. is convergent and therefore is absolutely convergent.

Consequently, the integral



CHECK YOUR PROGRESS 3

Q 1: Examine the convergence of the improper integral :

a) b)

c)

Q 2: Show that the following improper integrals are absolutely convergent

a) b)



14.8 LET US SUM UP

- An improper integral with finite interval of integration [a, b] with its integrand having a finite number of points of infinite discontinuity is said to be Improper integral of the first kind.
- An improper integral with infinite interval of integration and bounded integrand is said to be Improper integral of the second kind.
- An improper integral with infinite interval of integration with its integrand having a finite number of points of infinite discontinuity is said to be Improper integral of the mixed kind.
- For closed and bounded interval [a, b], let the left end point a be the only point of infinite discontinuity of a function f which is bounded and integrable on for every satisfying . Let , If exists (finitely) then the improper integral is said to be convergent and we write I where I is the limit.
  If
  is a said to be divergent.
- For closed and bounded interval [a, b], let the right end point b of the closed and bounded interval [a, b] be the only point of infinite discontinuity of a function f which is bounded and integrable on for every satisfying
  ,

If [ ] exists (finitely) then the improper integral [ ]

is a said to be convergent. If the limit be *l*, we write [ ] *1*.

- Let the end points a, b of the closed and bounded interval [a, b] be the only points of infinite discontinuity of a function f which is bounded and integrable on [ ] for every [ ] satisfying [ ], [ ]. Let c [ ](a, b). If the improper integrals [ ] [ ] be both convergent according to the definitions given above, then the improper integral [ ] and we write [ ]

- If the improper integrals [ ] all exist such that c_1, c_2, ...c_n are any finite number of points of discontinuity of f in [a, b], where [ ] [ ] [ ] then we say that [ ] [ ]

- A necessary and sufficient condition for the convergence of the improper integral [ ] is that there exists a positive real number k such that [ ] for every f satisfying [ ]

- Let a be the only point of infinite discontinuity of the functions f and g which are both integrable on [ ] [ ] and [ ] for all [ ]

(i)  is convergent  is convergent

(ii)  is divergent  is divergent

- The integral  is convergent if and only if 
- Let a be the only point of infinite discontinuity of a function f which is integrable on  for  and f (x) > 0 for all 
  If  *l* where *l* is a non-zero finite number, then the integral  is convergent if and only if 
- The improper integral 
  if 
- An absolutely convergent improper integral 
  the only point of infinite discontinuity of f in [a, b] and f is integrable on  for all  satisfying 
- Let a function f be integrable on [a, X] for all X > a and f (x) > 0 for all . A necessary and sufficient condition for the convergence of the improper integral 
  number k such that
   for all X > a
- Let the functions f and g be both integrable on [a, X] for all X > a and  for all 
  (i)  is convergent  is convergent
  (ii)  is divergent  is divergent.
- The integral , where a > 0, is convergent if and only if 
- Let f (x) > 0 for all . If  *l*, where *l* is a non-zero finite number, the improper integral 

only if

- A necessaryand sufficient condition for the convergence of the improper integral is that for a pre-assigned positive positive number X_0 such that

  <

  $_1$, $X_2 > X_0$.

- The improper integral

  if the integral

- Let improper integral

  the integral

- An absolutely convergent improper integral

  bounded and integrable on [a, X] for every X > a) is convergent.



14.9 ANSWERS TO CHECK YOUR PROGRESS

Check your progress 1

Ans to Q No 1: Since 0 is a point of infinite discontinuity of the integrand, so it is improper. The integrand is bounded and integrable on for all satisfying ,

Therefore the improper integral

.

Ans to Q No 2: Try Yourself.

Ans to Q No 3: Try Yourself.

Ans to Q No 4: Try Yourself.

Check your Progress 2

Ans to Q No 1: Try Yourself.

Ans to Q No 2: Try Yourself.

Ans to Q No 2: Try Yourself.

Check your Progress 3

Ans to Q No 1: Try Yourself.

Ans to Q No 2: Try Yourself.

14.10 FURTHER READING

1) Walter, R. Principles of Mathematical Analysis, 3rd edition , McGraw-Hill, 2017.
2) Terence T. Analysis II. Hindustan Book Agency, 2009.
3) Malik, S.C. and Arora, S. Mathematical Analysis. 2nd edition reprint. New Age International Publishers 2005.
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6) Royden, H.L. Real Analysis, Macmillan Pub, Co., Inc. 4th edition, New York, 1993.
7) Bartle, R.G. Sherbert, D.R. (2013). Introduction to real analysis. Hoboken, NJ : Wiley.
8) Narayan, S., Raisinghania, M.D. Elements of Real Analysis S. Chand & Company Pvt. Ltd. 2015.

14.11 MODEL QUESTIONS

Q 1: Determine the convergence of the improper integrals :

a)  b) 

c) 

Q 2: Test the convergene of the following integrals :

a)  b) 

c)  d) 

e)  f) 

Q 3: Examine the convergence of the improper integrals :

a)  b) 

c) 

Q 4: Show that the following improper integrals are absolutely convergent

a)  b) 

*** ***** ***



REFERENCES

1. Walter, R. *Principles of Mathematical Analysis*. 3rdedition, McGraw-Hill, 2017.
2. Terence T. *Analysis II*. Hindustan Book Agency, 2009.
3. Malik, S. C. and Arora, S. *Mathematical Analysis*. 2nd edition reprint. New Age International Publishers 2005.
4. Apostol, T. M. *Mathematical Analysis*. 2nd edition. *Wesley Publishing Co.* 2002.
5. Somasundram, D. and Chaudhary, B. *A First Course in Mathematical Analysis*. Narosa Publishing House, 1996.
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